The Primitive Groups of Class Twelve.

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The list of the primitive groups of the first thirteen classes prepared by Jordan has long been known to be defective in the case of class 12; in fact, Jordan states explicitly that the calculations for that case, having been gone through but once, may very well contain errors.* In spite of the considerable advances in substitution theory in recent years, the complete *à priori* determination of all the groups of this class still involves a good deal of labor. In order therefore not to intrude too far upon the space of this journal, the writer avoids a redetermination of the primitive groups of class 12 on less than 21 letters.† In a series of memoirs and in his lectures Professor Miller has gone over this ground without use of the list of the groups of class 12. He has thus checked the results of Miss Martin, of Jordan, and of Miss Bennett on the degrees 18, 19, and 20, respectively, according to which there is no primitive group of class 12 on either of these three degrees.

The method here followed will be much the same as that used by the author in treating the classes 6, 8, and 10, with the advantage of having at hand the processes employed in the paper entitled "On the Limit of the Degree of Primitive Groups." This last paper, as well as that "On Multiply Transitive Groups," and that "On the Order of Primitive Groups," will be used freely without specific reference.‡

In addition to the 25 primitive groups of class 12 of degree less than 18, there are four others, one of degree 27, two of degree 28, and one of degree 36.

* C. Jordan, *Comptes Rendus*, Vol. LXXV (1872), p. 1754.

† Miller, Degrees 13 and 14, *Quarterly Journal*, Vol. XXIX (1897), p. 224; Degree 15, *Proceedings of the London Mathematical Society*, Vol. XXVIII (1897), p. 533; Degree 16, American Journal of Mathematics, Vol. XX (1898), p. 229; Degree 17, *Quarterly Journal*, Vol. XXXI (1899), p. 49.

E. N. Martin, Degree 18, American Journal of Mathematics, Vol. XXIII (1901), p. 259.

E. R. Bennett, Degree 20, American Journal of Mathematics, Vol. XXXIV (1912), p. 1.

‡ Classes 6 and 8, American Journal of Mathematics, Vol. XXXII (1910), p. 235; Class 10, *op. cit.*, Vol. XXVIII (1906), p. 226; "On the Limit of the Degree of Primitive Groups," *Transactions of the American Mathematical Society*, Vol. XII (1911, p. 375); "On Multiply Transitive Groups," *op. cit.*, Vol. VII (1906), p. 499; "On the Order of Primitive Groups," *op. cit.*, Vol. X (1909), p. 247.

Only one of these is doubly transitive. It, and its maximal subgroup, which is primitive of degree 27, seem not to have been before noticed.

A regular group of degree and order 12 can not be multiply imprimitive in a sufficiently great number of ways to be a subgroup of a primitive group of higher degree than 20.

Let G contain the substitution

$$s_1 = a_1 a_2 a_3 \cdot b_1 b_2 b_3 \cdot c_1 c_2 c_3 \cdot d_1 d_2 d_3.$$

If it is impossible to say that among the totality of substitutions in G similar to s_1 there are two substitutions such that one connects cycles of the other and has at most one new letter in a cycle, we at any rate know that we can find two substitutions s_1 and s_2 of such a nature that $\{s_1, s_2\}$ has one imprimitive constituent on at least nine letters all of which are displaced by both s_1 and s_2.

If there are just two letters new to s_1 in one cycle of s_2, the group $H_2 \equiv \{s_1, s_2\}$ has an alternating constituent of order 60, which, however, can not be in isomorphism to a transitive group of degree 9.

Then let s_2 have a cycle of new letters $(\alpha\beta\gamma)$. At once, on forming the commutator,

$$s_1 s_2 = s_2 s_1.$$

Hence

$$s_2 = a_1 b_1 c_1 \cdot a_2 b_2 c_2 \cdot a_3 b_3 c_3 \cdot \alpha\beta\gamma.$$

Since the constituent in the letters $a_1, \ldots.$ is of degree greater than 4, there is a substitution s_3 similar to s_1, which replaces a_1 by a letter of another set, by the letter d_1, as we may assume without loss of generality, and which has at most one new letter to a cycle. The group H_2 can not be a subgroup of a transitive group of degree less than 20. To confirm this statement only the degrees 16 and 18 require examination. In the case of the degree 16, let δ be the letter fixed by H_2. Now the transitive group that is to include H_2 must be five-fold imprimitive in systems of four letters. But the only possible systems that can have δ in common are $d_1 d_2 d_3 \delta$ and $\alpha\beta\gamma\delta$. If the degree is 18, let δ_1, δ_2 and δ_3 be the three letters fixed by H_2. The only systems of two letters that can have δ_1 in common are $\delta_1\delta_2$ and $\delta_1\delta_3$, and the only system of three letters that can contain δ_1 is $\delta_1\delta_2\delta_3$. Then H_3 is intransitive. The constituent on the letters $\alpha, \ldots.$ is not of degree greater than 4, because the letters α, β, γ can not occur in more than one cycle of s_3. Since the other constituent can not be of degree 13, 14, or 15, the degrees of the two constituents are fixed as 12 and 4. Now H_4 must be an imprimitive group of degree 20, so that

$$s_4 = (a_1 \omega \varepsilon_1)(\text{——}\,\varepsilon_2)(\text{——}\,\varepsilon_3)(\text{——}\,\varepsilon_4),$$

where ω is one of the letters $\alpha, \beta, \gamma, \delta$ of H_3. If ω is δ, transform s_1 and s_2 by s_4; since both these transforms fix ε_1, and since s_4 can not replace all four letters a_2, a_3, b_1, c_1 by letters not included among these four, we see that ω can not be δ. Let ω be α; returning to s_3, that substitution has a cycle new to s_1 and we may assume $s_3 = (a_1 - d_1) \ldots . (- \delta)$. Now if s_1 connects two cycles of s_3, s_1 has in some cycle two letters new to s_3. Then $s_1^{-1} s_3^{-1} s_1 s_3$, or else $s_1 s_3^{-1} s_1^{-1} s_3$, is of degree less than 12. If s_1 does not connect two cycles of s_3, the first cycle $(a_1 - d_1)$ of s_3 is the only cycle of s_3 which contains any one of the six letters a_1, a_2, a_3, d_1, d_2, or d_3, so that s_3 fixes two a's or two d's, and again the degree of $s_1^2 s_3^2 s_1 s_3$ (or $s_1 s_3^2 s_1^2 s_3$) is less than 12. Then s_3 and s_1 are commutative:

$$s_3 = (a_1 b\, d_1)(a_2 b\, d_1)(a_3 b\, d_3)(-- \delta).$$

If s_3 were to connect two cycles of s_2, there would be two letters d in one cycle of s_3, hence

$$s_3 = (a_1 b_1 d_1)(a_2 b_2 d_2)(a_3 b_3 d_3)(-- \delta).$$

Since $s_4^2 s_1 s_4$ fixes ε_1, s_4 can not replace a_2 or a_3 by any letter of the set $a_1 \ldots .$ of H_3. If $s_4 = (a_1 \alpha \varepsilon_1)(- c_1 \varepsilon_2)(-- \varepsilon_3)(-- \varepsilon_4)$, $s_4^2 s_3 s_4$ fixes ε_2, and therefore s_4 can not replace b_1 or d_1 by any letter of the first set $a_1 \ldots .$ of H_3. This is in contradiction with the preceding statement regarding a_2 and a_3. Likewise $s_4 = (- b_1 \varepsilon_2) \ldots .$ is impossible because of the transform of $s_2 s_3 s_2^2$ by s_4, which fixes ε_2 and requires that s_4 replace c_1 and d_1 by letters not in the set $a_1 \ldots .$ of H_3. But we can show that s_4 must replace b_1 or c_1 by a letter ε. One of the substitutions $s_4^2 s_1 s_4$ or $s_4^2 s_2 s_4$ has in its first cycle the letter α and one of the letters a_2, a_3, b_1 or c_1. This transform (σ) must displace the four letters $\varepsilon_1, \ldots, \varepsilon_4$. If two of the letters $\varepsilon_1, \ldots .$ are in the same cycle, the group $\{H_3, \sigma^{-1} H_3 \sigma\}$ has a generator joining the two sets of H_3, with less than four new letters ε. Then in σ the four letters $\varepsilon_1, \ldots .$ are in different cycles and σ is the transform of s_2 by s_4. Hence s_4 replaces b_1 or c_1 by a letter ε_2.

Then there are in G some two substitutions s_1 and s_2 of degree 12 and order 3, such that s_2 connects cycles of s_1 and has at most one new letter in a cycle. Since $H_2 \equiv \{s_1, s_2\}$ has a set of at least six letters, we form the successive groups H_3, H_4, enlarging the set a of H_i (say) by union with some other set of H_i by means of a substitution s_{i+1} which has at most one new letter in any cycle $(i = 2, 3)$.

If H_4 is the first transitive group of the series, H_2 has three sets of degrees ranging from 6,3,3 to 8,4,4. The constituent of highest degree is in all cases in simple isomorphism (to keep up the class) with one of the others, which are of order 3 or 12. But this is impossible. If H_i $(i = 2$ or $3)$ is transitive, the

only degrees to be considered are 16, 18 and 20. Let H_i be of degree 16. Then it is at least five-fold imprimitive in systems of four letters. But if α is one of the new letters $\alpha\beta\gamma\delta$, s_1 transforms the systems to which α belongs into themselves and therefore they are only

$$a_1 a_2 a_3 \alpha, \quad b_1 b_2 b_3 \alpha, \quad c_1 c_2 c_3 \alpha, \quad d_1 d_2 d_3 \alpha, \quad \beta\gamma\delta\alpha.$$

But consider s_2 (or s_3), which displaces all the letters $\alpha\beta\gamma\delta$, and suppose α chosen so that in the cycle of s_2 (or s_3) with α the two remaining letters are from different cycles of s_1. This is legitimate since some generator must permute systems. Now s_2 (or s_3) can transform into itself a system to which a letter left fixed (say a_1) belongs only if the two other letters making up that system (a_2, a_3) are in the same cycle with α. Then the degree of H_i is greater than 16 and i is equal to 3 only. Let the degree of H_3 be 18. The systems consist of two or three letters. Letters in H_{i+1} and not in H_i $(i = 1, 2)$ can not be in a system with letters of H_i, because of the smallness of the systems. For some value of i, H_{i+1} introduces at most three new letters. Then one of them is not in more than one system of three letters, nor in more than two systems of two letters. Let H_3 be of degree 20. It is imprimitive. The systems are of two or four letters. Systems of two letters can not be permuted according to a primitive group, because of the substitutions of order 3 and degree 12 which permute systems, and cause the group in the systems to be of class 6 or less and hence alternating. This brings in a substitution of order 7. Hence there are in all cases systems of four letters permuted according to G_{60}^5. Now H_2 has just two sets of letters and is of degree 16. Suppose first that one constituent is of degree 12 and the other of degree 4. Since the latter is generated by two circular substitutions of order 3, it is of order 12. The isomorphism between the two constituents can not be simple. The head of degree and class 12 can only be of order 3, as is seen by recalling that the average number of letters in the intransitive group H_2 is 14. Then H_2 is of order 36. Assume the substitution of the head to be

$$t = a_1 b_1 c_1 \cdot a_2 b_2 c_2 \cdot a_3 b_3 c_3 \cdot \alpha\beta\gamma$$

and

$$s_1 = a_1 a_2 a_3 \cdot b_1 b_2 b_3 \cdot c_1 c_2 c_3 \cdot d_1 d_2 d_3 .$$

Now t is in each of the four subgroups of order 9 of H_2, and H_2 is doubly transitive in the systems. Hence

$$s_2 = a_1 \alpha b_2 \cdot b_1 \beta c_2 \cdot c_1 \gamma a_2 \cdot d d \delta,$$

or

$$s_2' = (a_1 \alpha c_2) \ldots\ldots$$

But since $\{s_1, s'_2\}$ is the transform of $\{s_1, s_2\}$ by $b_1 c_1 \cdot b_2 c_2 \cdot b_3 c_3 \cdot \beta\gamma$, s'_2 may be dropped. Now

$$s_1 s_2 = (a_1 c_1 b_1)(a_2 a_3 \alpha b_2 b_3 \beta c_2 c_3 \gamma)(\ldots\ldots),$$

and $(s_1 s_2)^3$ is of degree 9, thus reducing the class of H_2. Let H_2 have two unconnected sets of eight letters each. The group in the systems is again G_{12}^4. The isomorphism between the two groups is simple. The two constituents permute corresponding systems of two letters each. Consider one constituent: it is generated by

$$\bar{s}_1 = a_1 a_2 a_3 \cdot b_1 b_2 b_3$$

and

$$\bar{s}_2 = a_1 \alpha b_2 \cdot b_1 \beta a_2$$

uniquely;

$$(\bar{s}_1 \bar{s}_2)^3 = a_1 b_1 \cdot a_2 b_2 \cdot a_3 b_3 \cdot \alpha \beta$$

is an invariant operator. The group $\bar{H}_2 \equiv \{\bar{s}_1, \bar{s}_2\}$ is of order 24. Then H_2^{16} has an invariant substitution t of order 2 and degree 16. Let $\alpha', \beta', \gamma', \delta'$ be the four letters of H_3 new to H_2. Since H_3 is triply transitive in its systems, we may say that s_3 permutes the systems $a_1 b_1 c_1 d_1$, $\alpha\beta\gamma\delta$, $\alpha'\beta'\gamma'\delta'$. Then t and $s_3^2 t s_3$ have in common four cycles in the letters of the two other systems $a_2 b_2 c_2 d_2$, $a_3 b_3 c_3 d_3$, so that $t s_3^2 t s_3$ is of degree 12. Again, $t s_3^2 t s_3$ and $s_1^2 t s_3^2 t s_3 s_1$ have in common four cycles of the Greek letters, so that $t s_3^2 t s_3 s_1^2 t s_3^2 t s_3 s_1$ is of degree 8.

In conclusion, no primitive group of degree exceeding 20 contains a substitution of order 3 and degree 12.

There is in G a substitution

$$s_1 = a_1 a_2 \cdot b_1 b_2 \cdot c_1 c_2 \cdot d_1 d_2 \cdot e_1 e_2 \cdot f_1 f_2.$$

We shall first set up a list of the diedral rotation groups which two substitutions similar to s_1 may generate and which G may possibly include. Now G can contain no substitution of order 3 and degree 12, no substitution of order 5 and degree less than 20, no substitution of order 7, 11, 13 or 17 and of degree less than 21. Nor has it a circular substitution of degree 18 or less. The group generated by all the substitutions of G which are similar to s_1 is positive. Then the only types of substitutions of degree less than 19 in $\{s_1, \ldots\ldots\}$ are

$$(a_1 a_2 a_3 \ldots\ldots a_{16})(b_1 b_2), \qquad (a_1 a_2 \ldots\ldots a_9)(b_1 b_2 \ldots\ldots b_9),$$
$$(a_1 a_2 \ldots\ldots a_{12})(b_1 b_2 \ldots\ldots b_6), \qquad (a_1 a_2 \ldots\ldots a_{12})(b_1 b_2 b_3)(c_1 c_2),$$

and their powers, as can easily be verified directly. And the only types that

can occur in the diedral rotation groups generated by two substitutions similar to s_1 are powers of

$$a_1 a_2 a_3 a_4 \cdot b_1 b_2 b_3 b_4 \cdot c_1 c_2 c_3 c_4 \cdot d_1 d_2,$$
$$a_1 a_2 a_3 a_4 a_5 a_6 \cdot b_1 b_2 b_3 b_4 b_5 b_6 \cdot c_1 c_2 c_3,$$
$$a_1 a_2 a_3 a_4 \cdot b_1 b_2 b_3 b_4 \cdot c_1 c_2 c_3 c_4 \cdot d_1 d_2 d_3 d_4,$$
$$a_1 a_2 a_3 a_4 \cdot b_1 b_2 b_3 b_4 \cdot c_1 c_2 c_3 c_4 \cdot d_1 d_2 \cdot e_1 e_2 \cdot f_1 f_2,$$
$$a_1 a_2 a_3 \cdot b_1 b_2 b_3 \cdot c_1 c_2 c_3 \cdot d_1 d_2 d_3 \cdot e_1 e_2 e_3 \,.\, f_1 f_2 f_3.$$

In this statement only two points seem to require special mention. First, if the diedral group in question is of degree greater than 18, it is Abelian. Suppose $\{s_1, s_2\}$ of degree 19 and non-Abelian. Since s_2 now has at least one cycle of letters new to s_1, and vice-versa, $(s_1 s_2)^2$ is of degree 15 at most. If the degree is 15, $s_1 s_2$ has five cycles of three letters each, so that $(s_1 s_2)^3$ is of degree 4. Now $(s_1 s_2)^2$ must displace an odd number of letters and is not of degree 13. Suppose $\{s_1, s_2\}$ of degree 20. If the product $s_1 s_2$ has four transpositions (cycles of order 2), $(s_1 s_2)^2$ is of order 3 and degree 12. The class is lower than 12 if there are more than four cycles of order 2 in the product, and so many of these cycles certainly occur. If $\{s_1, s_2\}$ is of degree 21 or more, the commutator $(s_1 s_2)^2$ is of degree 9 at most. In the second place, it may be asked why the type $(a_1 a_2 \ldots . a_8)(b_1 b_2 \ldots . b_8)$ is excluded from the second list. Since the two constituent diedral groups are generated by an odd and an even substitution of order 2, substitutions of degree 6 do not correspond to substitutions of degree 6, but to substitutions of degree 8, so that this product belongs to a diedral group of class 14.

From these substitutions we now construct the following list of diedral groups. Each is generated by

$$a_1 a_2 \cdot b_1 b_2 \cdot c_1 c_2 \cdot d_1 d_2 \cdot e_1 e_2 \cdot f_1 f_2$$

and a substitution s_2 which is written out along with a numbered symbol "D" to distinguish the group.

$$D_1 : \quad a_1 b_1 \cdot a_2 b_2 \cdot c_1 d_1 \cdot c_2 d_2 \cdot e_1 f_1 \cdot e_2 f_2 ;$$
$$D_2 : \quad a_1 b_1 \cdot a_2 b_2 \cdot c_1 d_1 \cdot c_2 d_2 \cdot e_1 e_2 \cdot \alpha_1 \alpha_2 ;$$
$$D_3 : \quad a_1 b_1 \cdot a_2 b_2 \cdot c_1 d_1 \cdot c_2 d_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2 ;$$
$$D_4 : \quad a_1 b_1 \cdot a_2 b_2 \cdot c_1 c_2 \cdot d_1 d_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2 ;$$
$$D_5 : \quad a_1 b_1 \cdot a_2 b_2 \cdot c_1 c_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2 \cdot \gamma_1 \gamma_2 ;$$
$$D_6 : \quad a_1 b_1 \cdot a_2 b_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2 \cdot \gamma_1 \gamma_2 \cdot \delta_1 \delta_2 ;$$
$$D_7 : \quad a_1 a_2 \cdot b_1 b_2 \cdot c_1 c_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2 \cdot \gamma_1 \gamma_2 ;$$
$$D_8 : \quad a_1 a_2 \cdot b_1 b_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2 \cdot \gamma_1 \gamma_2 \cdot \delta_1 \delta_2 ;$$
$$D_9 : \quad a_1 a_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2 \cdot \gamma_1 \gamma_2 \cdot \delta_1 \delta_2 \cdot \varepsilon_1 \varepsilon_2 ;$$

$$
\begin{aligned}
D_{10}:&\quad \alpha_1\alpha_2\cdot\beta_1\beta_2\cdot\gamma_1\gamma_2\cdot\delta_1\delta_2\cdot\varepsilon_1\varepsilon_2\cdot\zeta_1\zeta_2\,;\\
D_{11}:&\quad a_1b_2\cdot b_1c_2\cdot c_1d_2\cdot d_1a_2\cdot e_1\alpha_1\cdot e_2\alpha_2\,;\\
D_{12}:&\quad a_1b_2\cdot b_1c_2\cdot c_1d_2\cdot d_1a_2\cdot e_1f_1\cdot\alpha_1\alpha_2\,;\\
D_{13}:&\quad a_1b_1\cdot a_2c_1\cdot b_2c_2\cdot d_1\delta\cdot e_1\varepsilon\cdot f_1\zeta\,;\\
D_{14}:&\quad a_1b_1\cdot a_2c_1\cdot d_1e_1\cdot d_2\alpha_1\cdot e_2\alpha_2\cdot f_1\zeta\,;\\
D_{15}:&\quad a_1b_1\cdot c_1d_1\cdot e_1\varepsilon_1\cdot e_2\varepsilon_2\cdot f_1\zeta_1\cdot f_2\zeta_2\,;\\
D_{16}:&\quad a_1b_1\cdot c_1c_2\,.\,d_1\delta_1\cdot d_2\delta_2\cdot e_1\varepsilon_1\cdot e_2\varepsilon_2\,;\\
D_{17}:&\quad a_1b_1\cdot c_1d_1\cdot e_1e_2\cdot f_1\zeta_1\cdot f_2\zeta_2\cdot\eta_1\eta_2\,;\\
D_{18}:&\quad a_1\alpha_1\cdot a_2\alpha_2\cdot b_1\beta_1\cdot b_2\beta_2\cdot c_1\gamma_1\cdot c_2\gamma_2\,;\\
D_{19}:&\quad a_1b_1\cdot c_1d_1\cdot e_1f_1\cdot\alpha_1\alpha_2\cdot\beta_1\beta_2\cdot\gamma_1\gamma_2\,;\\
D_{20}:&\quad a_1b_1\cdot c_1\gamma_1\cdot c_2\gamma_2\cdot d_1\delta_1\cdot d_2\delta_2\cdot\varepsilon_1\varepsilon_2\,;\\
D_{21}:&\quad a_1b_1\cdot c_1d_1\cdot e_1\alpha_1\cdot e_2\alpha_2\cdot\beta_1\beta_2\cdot\gamma_1\gamma_2;\\
D_{22}:&\quad a_1\alpha\cdot b_1\beta\cdot c_1\gamma\cdot d_1\delta\cdot e_1\varepsilon\cdot f_1f_2\,;\\
D_{23}:&\quad a_1\alpha\cdot b_1\beta\cdot c_1\gamma\cdot d_1\delta\cdot e_1\varepsilon\cdot f_1\zeta\,.
\end{aligned}
$$

Of these groups the first ten only are Abelian. The two groups D_{11} and D_{12} are the same, the order in which the two generators are taken being reversed. The same is true of the three pairs: D_{16} and D_{17}; D_{18} and D_{19}; D_{20} and D_{21}. We shall show that of all these groups only D_6, D_{10} and D_{23} are actually subgroups of a primitive group G of class 12 and of degree greater than 20.

The first of these groups which we take up for discussion is D_{14}. Those substitutions of D_{14} of which we shall make use are:

$$
\begin{aligned}
s_1 &= a_1a_2\cdot b_1b_2\cdot c_1c_2\cdot d_1d_2\cdot e_1e_2\cdot f_1f_2,\\
s_2 &= a_1b_1\cdot a_2c_1\cdot d_1e_1\cdot d_2\alpha_1\cdot e_2\alpha_2\cdot f_1\zeta,\\
t_1 &= a_2b_2\cdot a_1c_2\cdot d_2e_2\cdot d_1\alpha_1\cdot e_1\alpha_2\cdot f_2\zeta,\\
t_2 &= b_1c_2\cdot b_2c_1\cdot d_1e_2\cdot d_2e_1\cdot\alpha_1\alpha_2\cdot f_1f_2,\\
t_3 &= a_1b_2\cdot a_2c_2\cdot b_1c_1\cdot d_1\alpha_2\cdot e_1\alpha_1\cdot f_2\zeta,\\
t_4 &= a_1c_1\cdot a_2b_1\cdot b_2c_2\cdot d_2\alpha_2\cdot e_2\alpha_1\cdot f_1\zeta,\\
t_5 &= a_1a_2\cdot b_1c_1\cdot b_2c_2\cdot d_1e_1\cdot d_2e_2\cdot\alpha_1\alpha_2,\\
(s_1s_2)^2 &= a_1c_2b_1\cdot a_2b_2c_1\cdot d_1d_2\alpha_2\cdot e_1e_2\alpha_1\cdot f_2f_1\zeta\,.
\end{aligned}
$$

To the set $a_1a_2b_1b_2c_1c_2$ is joined another set of D_{14} by a substitution s_3, similar to s_1, with at most one new letter in any cycle. The degree of the resulting transitive constituent is 9, 12, 16, 18 or 20. If the degree is 18 or 20, since D_{14} can contribute at most 12 letters to a transitive constituent of an intransitive $H_3\equiv\{D_{14},\, s_3\}$, the group H_3 is transitive. However, if s_3 introduces five new letters it can connect only two sets of D_{14}. Hence the degree 20 is not to be considered. In this connection it is well to remark that D_{14} can not be contained in a transitive subgroup H_i of $\{s_1, \ldots\}$ of degree less than 20. First,

such a subgroup being generated by $s_1, s_2, \ldots.$ can have no system of imprimitivity of 8 or 9 letters. Second, it can have no system of two letters because of $(s_1 s_2)^2$. Third, because of $(s_1 s_2)^2$, s_1, s_2, jointly, a system of three, four or six letters involving a common letter can not be chosen in more than one way,

Hence H_3 is intransitive, and we assume for the moment that the larger constituent is of degree 16. Let

$$s_3 = (a_1 d_1)(\beta_1 -)(\beta_2 -)(\beta_3 -)(\beta_4 -)(- -).$$

Since s_3 generates with each of the substitutions s_1, s_2, t_1, t_3, t_5 a group of degree 16 or 18, s_3 fixes $a_2, d_2, b_1, e_1, c_2, \alpha_1, b_2, \alpha_2$, and only c_1 and e_2 remain to fill the four places with $\beta_1, \ldots., \beta_4$. Let

$$s_3' = (a_1 \alpha_1)(\beta_1 -)(\beta_2 -)(\beta_3 -)(\beta_4 -)(- -).$$

For the same reason as before s_3' fixes $b_1, d_2, c_2, d_1, b_2, e_1, c_1, e_2, a_2, \alpha_2$. The substitutions $d_1 e_1 \cdot d_2 e_2 \cdot \alpha_1 \alpha_2$ and $b_1 c_2 \cdot b_2 c_1 \cdot d_1 d_2 \cdot e_1 e_2 \cdot f_1 f_2$ transform D_{14} into itself and consequently $s_3 = (a_1 d_1) \ldots.$ and $s_3' = (a_1 \alpha_1) \ldots.$ carry with them $s_3'' = (a_1 d_2) \ldots.$, etc. Likewise $(a_1 f_1) \ldots.$ and $(a_1 f_2) \ldots.$ are conjugate. However, $(a_1 f_1) \ldots.$ can not in this case give a constituent of degree 16.

Assume the larger constituent to be of degree 12. Let

$$s_3 = (a_1 f_1)(\beta_1 -)(\beta_2 -)(\beta_3 -)(- -)(- -),$$

where no letter $d_1, d_2, e_1, e_2, \alpha_1, \alpha_2$ is replaced by β_1, β_2 or β_3. The second constituent is of degree 6 or 8.

Suppose the second constituent of degree 8:

$$s_3 = (a_1 f_1)(\beta_1 -)(\beta_2 -)(\beta_3 -)(\gamma_1 -)(\gamma_2 -).$$

As before, s_3 fixes $a_2, f_2, b_1, \zeta, c_1$, leaving only b_2 and c_2 to fill the three places with β_1, β_2 and β_3. If

$$s_3 = (a_1 \zeta_1)(\beta_1 -)(\beta_2 -)(\beta_3 -)(\gamma_1 -)(\gamma_2 -),$$

s_3 fixes $b_1, f_1, c_2, f_2, b_2, c_1$.

Suppose the second constituent of degree 6, and let

$$s_3 = (a_1 f_1) \ldots..$$

Neither $\{s_1, s_3\}$ nor $\{t_4, s_3\}$ can be of degree 15. For were they of degree 15, the letters $d_1 d_2 e_1 e_2 \alpha_1 \alpha_2$ would form one of the constituents of degree 6 in $\{s_1, s_3\}$ and $\{t_4, s_3\}$. Not being united to new letters they can not form a set of three letters. But since s_1 and t_4 have only two cycles in these letters (as has s_3), this is not possible. If $\{s_2, s_3\}$ is of degree 15, s_3 must displace either c_1 or ζ, but from $\{t_4, s_3\}$, s_3 fixes both c_1 and ζ. Hence a_2, f_2, b_1, ζ and c_1 are fixed by s_3, leaving only b_2 and c_2 for the cycles with β_1, β_2 and β_3. Let

$$s_3 = (a_1 \zeta) \ldots..$$

Since, as before, $\{t_3, s_3\}$ and $\{t_4, s_3\}$ can not be of degree 15, s_3 fixes f_1 and f_2, and therefore neither $\{s_2, s_3\}$ nor $\{t_1, s_3\}$ is of degree 15. For since s_3 has only four letters of one constituent of degree 6, it must have six letters of the other. Then $b_1, f_1, c_2, f_2, b_2, c_1$ are fixed, leaving only a_2 free to enter s_3.

If $s_3 = (a_1 d_1) \ldots.$, the second set is of degree 3, 4 or 6. Let

$$s_3 = (a_1 d_1)(--)(--)(f_1 \beta_1)(f_2 \beta_2)(\zeta \beta_3).$$

From $\{s_1, s_3\}$, s_3 fixes a_2 and d_2. Hence $\{t_5, s_3\}$ is such a group as D_{19}, and s_3 displaces one but not both the letters α_1 and α_2. But this would make $\{s_1, s_3\}$ of degree 17. Let

$$s_3 = (a_1 \alpha_1)(--)(--)(f_1 \beta_1)(f_2 \beta_2)(\zeta \beta_3).$$

From $\{s_1, s_3\}$, s_3 must have the cycle $(a_2 \alpha_2)$. Now $\{s_2, s_3\}$ is of degree 16 or 18, so that s_3 fixes b_1, d_2, c_1, and e_2. From $\{t_1, s_3\}$, s_3 fixes c_2, d_1, b_2, and e_1, so that no letter is left for the third cycle of s_3. Now let the second set have just one new letter β. This constituent of degree 4 is of order 24. Since H_3 contains no substitution of degree 12 connecting the set $a_1 \ldots.$ and the set $d_1 \ldots.$ of D_{14} without a new letter β, the constituent of degree 12 is of order 48, and t_5 is invariant in H_3. Then

$$s_3 = (a_1 d_1)(a_2 e_1) \ldots, (\beta_1 -).$$

Now ζ is new to β_1, f_2 to s_2, f_1 to t_1, and with s_1, s_2, and t_1, severally, s_3 generates a group that has a constituent on six or more letters. In one of the groups $\{s_1, s_3\}$, $\{s_2, s_3\}$, $\{t_1, s_3\}$, s_3 has a cycle of "new letters," either $(f_1 \beta)$, $(f_2 \beta)$ or $(\zeta \beta)$. That group can not be the D_{12} because s_3 can not join the cycle $(f_1 f_2)$ to another cycle of s_1; similar statements hold for $(f_1 \zeta)$, $(f_2 \zeta)$ and s_2, t_1, respectively. Hence that group can only be D_{17}. If

$$s_3 = a_1 d_1 \cdot a_2 e_1 \cdot \zeta \beta \ldots.,$$

$\{s_1, s_3\}$ is not D_{17}. Nor if

$$s_3 = a_1 d_1 \cdot a_2 e_1 \cdot f_2 \beta \ldots.,$$

can $\{s_2, s_3\}$ be D_{17}. Then

$$s_3 = a_1 d_1 \cdot a_2 e_1 \cdot f_1 \beta \ldots.,$$

and fixes c_2, α_1, b_2, α_2. Now $\{s_1, s_3\}$ is of degree 14, that is, it is D_{11}, and therefore

$$s_3 = (a_1 d_1)(a_2 e_1)(f_1 \beta)(f_2 \zeta)(e_2 -)(d_2 -);$$

the two undetermined letters must be from the same cycle of s_1, but because $\{t_1, s_3\}$ is of degree 16, those two letters are b_1 and c_1. Let

$$s_3 = (a_1 \alpha_1)(a_2 \alpha_2) \ldots. (\beta -).$$

Since $(a_1 a_2)$ is a cycle of s_1, $\{s_1, s_3\}$ is not of degree 15, and therefore s_3 displaces ζ. Let us try

$$s_3 = a_1 \alpha_1 \cdot a_2 \alpha_2 \cdot f_1 \zeta \cdot f_2 \beta \ldots\ldots$$

To this is conjugate $a_1 \alpha_1 \cdot a_2 \alpha_2 \cdot f_2 \zeta \cdot f_1 \beta \ldots\ldots$ under $b_1 c_2 \cdot b_2 c_1 \cdot b_1 d_2 \cdot e_1 e_2 \cdot f_1 f_2$. Now $\{s_2, s_3\}$, as above, is of degree 16, and hence s_3 fixes b_1, d_2, c_1, e_2. It has d_1 and e_1 in different cycles, accompanied by the only letters we have new to s_2: b_2 and c_2. Then

$$s_3' = a_1 \alpha_1 \cdot a_2 \alpha_2 \cdot f_1 \zeta \cdot f_2 \beta \cdot d_1 b_2 \cdot e_1 c_2,$$

or

$$s_3'' = a_1 \alpha_1 \cdot a_2 \alpha_2 \cdot f_1 \zeta \cdot f_2 \beta \cdot d_1 c_2 \cdot e_1 b_2.$$

Before further study of s_3' and s_3'', let us consider

$$s_3 = (a_1 \alpha_1)(a_2 \alpha_2)(\zeta \beta) \ldots\ldots$$

Since $\{s_1, s_3\}$ is D_{17}, s_1 and s_3 have a cycle in common. If this common cycle is not $(f_1 f_2)$, s_3 replaces an f by a letter of another cycle of s_1. In the two remaining cycles we have one letter each from the cycles $(b_1 b_2)$, $(c_1 c_2)$, $(d_1 d_2)$, $e_1 e_2)$. Now $\{s_2, s_3\}$ can not be of degree 15. If $\{s_2, s_3\}$ is of degree 14, s_3 displaces b_1, c_1, d_2, e_2. If $\{s_2, s_3\}$ is of degree 16, s_3 displaces b_2, c_2, d_1, e_1. We may have

$$s_3''' = a_1 \alpha_1 \cdot a_2 \alpha_2 \cdot f_1 f_2 \cdot \zeta \beta \cdot b_1 c_1 \,.\, d_2 e_2,$$
$$s_3^{\mathrm{IV}} = a_1 \alpha_1 \cdot a_2 \alpha_2 \cdot f_1 f_2 \cdot \zeta \beta \cdot b_1 d_2 \cdot c_1 e_2,$$
$$s_3^{\mathrm{V}} = a_1 \alpha_1 \cdot a_2 \alpha_2 \cdot f_1 f_2 \cdot \zeta \beta \cdot b_1 e_2 \,.\, c_1 d_2,$$

and their conjugates under $b_1 c_2 \cdot b_2 c_1 \cdot d_1 d_2 \cdot e_1 e_2 \cdot f_1 f_2$.

Now pass to H_4; we have only to consider the degrees 20 and 21. Let

$$s_4 = (a_1 f_1)(\gamma_1 -)(\gamma_2 -)(\gamma_3 -)(\gamma_4 -)(\gamma_5 -).$$

From $\{s_1, s_4\}$, $\{s_2, s_4\}$, $\{t_4, s_4\}$, s_4 fixes $a_2, f_2, b_1, \zeta, c_1$. Now $\{t_5, s_4\}$ is of degree 18; it can not be D_{18} because a_2 is fixed by s_4, nor can it be D_{28} because both b_1 and c_1 are fixed. These are the only possibilities. Let

$$s_4 = (a_1 \zeta)(\gamma_1 -)(\gamma_2 -)(\gamma_3 -)(\gamma_4 -)(\gamma_5 -).$$

This s_4 fixes $b_1, f_1, c_2, f_2, b_2, c_1$. Because of $\{s_1, s_4\}$,

$$s_4 = a_1 \zeta \cdot a_2 \gamma_1 \cdot d_1 \gamma_2 \cdot d_2 \gamma_3 \cdot e_1 \gamma_4 \cdot e_2 \gamma_5;$$

but thus s_4 has eight letters new to t_3 and is not commutative with it. Let

$$s_4 = (a_1 \beta)(\gamma_1 -)(\gamma_2 -)(\gamma_3 -)(\gamma_4 -)(\gamma_5 -).$$

Here $\{t_2, s_4\}$ is of higher degree than 18, and non-Abelian.

Next let H_4 be of degree 20, and let

$$s_4 = (a_1 f_1)(--)(\gamma_1 -)(\gamma_2 -)(\gamma_3 -)(\gamma_4 -).$$

The groups $\{s_1, s_4\}$, $\{s_2, s_4\}$, $\{t_4, s_4\}$, $\{s_3, s_4\}$ (s_3 may be any one of the five substitutions $s'_3, \ldots$ given above) are of degree 16 or 18. Hence s_4 leaves fixed $a_2, f_2, b_1, \zeta, c_1, \alpha_1$. Now $\{t_2, s_4\}$ is of degree 17 and hence is of order 6; $\{t_5, s_4\}$ is also of degree 17. But s_4 can not have a cycle in common with both t_2 and t_5. Let

$$s_4 = (a_1 \zeta)(--)(\gamma_1 -)(\gamma_2 -)(\gamma_3 -)(\gamma_4 -).$$

This substitution fixes $b_1, f_1, c_2, f_2, b_2, c_1, \alpha$. The diedral group $\{s_1, s_4\}$ is D_{18}, so that s_4 displaces a_2. This fact makes $\{t_2, s_4\}$ impossible. Let

$$s_4 = (a_1 \beta)(--)(\gamma_1 -)(\gamma_2 -)(\gamma_3 -)(\gamma_4 -).$$

From $\{t_2, s_4\}$ it is clear that s_4 fixes ζ and a_2. Since the products $s_1 s_4$ and $t_5 s_4$ have the cycle $(a_1 a_2 \beta)$, they are of order 3. Hence s_4 displaces at least one f and at least one α, so that $\{s_1, s_4\}$ and $\{t_5, s_4\}$ are each of degree 18. Further, s_4 fixes one letter from each cycle of s_1 and t_5. Then $\{t_2, s_4\}$ is D_{20}, and since from $\{s_3, s_4\}$ α_1 is seen to be fixed by s_4, we have

$$s_4 = (a_1 \beta)(f_2 \alpha_2)(\gamma_1 -)(\gamma_2 -)(\gamma_3 -)(\gamma_4 -)(\alpha_1)(f_1)(a_2)(\zeta),$$

or else $(a_1 \beta)(f_1 \alpha_2) \ldots$, which we transform into $s_4 = (f_2 \alpha_2) \ldots$ by

$$b_1 c_2 \cdot b_2 c_1 \cdot d_1 d_2 \cdot e_1 e_2 \cdot f_1 f_2.$$

From $\{t_1, s_4\}$, s_4 fixes e_1, and from $\{t_3, s_4\}$, s_4 fixes d_1, so that

$$s_4 = (a_1 \beta)(f_2 \alpha_2)(d_2 \gamma_1)(e_2 \gamma_2) \ldots,$$

inconsistent with the remark that $t_5 s_4$ is of order 3.

While the larger constituent of H_3 is of degree 12, the other may be of degree 3. Pass at once to H_4. Let

$$s_4 = (a_1 f_1)(\gamma_1 -)(\gamma_2 -)(\gamma_3 -)(\gamma_4 -)(\gamma_5 -).$$

Now s_4 fixes $a_2, f_2, b_1, \zeta, c_1$; the product $t_5 s_4 = (a_1 a_2 f_1) \ldots$ is of order 3 and of degree 18, which is not possible since both b_1 and c_1 are fixed. Let

$$s_4 = (a_1 \zeta)(\gamma_1 -)(\gamma_2 -)(\gamma_3 -)(\gamma_4 -)(\gamma_5 -).$$

Here s_4 fixes $b_1, f_1, c_2, f_2, b_2, c_1$. From $\{s_1, s_4\}$ we have

$$s_4 = a_1 \zeta \cdot a_2 \gamma_1 \cdot d_1 \gamma_2 \cdot d_2 \gamma_3 \cdot e_1 \gamma_4 \cdot e_2 \gamma_5,$$

but now $\{t_2, s_4\}$ is impossible.

We return to D_{14} and seek s_3 under the assumption that H_3 has a constituent of degree 9. The other set has 6, 8, or 9 letters. First suppose that there are two constituents of degree 9. Let

$$s_3 = (a_1 f_1)(--)(--)(\beta_1 -)(\beta_2 -)(\beta_3 -).$$

With $\beta_1, \ldots$ are united only letters $d_1, d_2, e_1, e_2, \alpha_1, \alpha_2$. Since the letters with $\beta_1, \beta_2, \beta_3$ can not come from three different cycles of s_1 or t_4, the groups

$\{s_1, s_3\}$ and $\{t_4, s_3\}$ are not of degree 15. Then s_3 fixes a_2, f_2, c_1, ζ. Because ζ and f_2 are fixed, both $\{s_1, s_3\}$ and $\{t_4, s_3\}$ are of degree 16. Then s_3 fixes α_1 or α_2, and d_1 or e_1. It follows that s_3 displaces b_1, b_2, d_2, e_2 in addition to an α and one of the letters d_1, e_1. But s_3 must displace an odd number of the letters $d_1, d_2, e_1, e_2, \alpha_1, \alpha_2$. Let

$$s_3 = (a_1 \zeta)(--)(--)(\beta_1 -)(\beta_2 -)(\beta_3 -).$$

Again $\{t_3, s_3\}$ and $\{t_4, s_3\}$ are not of degree 15; hence s_3 fixes b_2, c_1, f_1, f_2, in consequence of which each group is of degree 16. Then s_3 fixes d_1 or e_1, and fixes also one of the letters d_2 and e_2. Hence the remaining five letters displaced by s_3 are $a_2, b_1, c_2, \alpha_1, \alpha_2$. But as before, this gives an even number of letters $d_1, d_2, e_1, e_2, \alpha_1, \alpha_2$ in s_3.

Suppose that there are not more than two new letters in the second constituent of H_3. The isomorphism between the two constituents is simple. If a transitive group of degree less than 9 has a subgroup of order 9, it contains a cycle of three letters. Then to this circular substitution of order 3 there corresponds in the other constituent a substitution of degree 9. Since any substitution of degree 12 in G is regular, here is a substitution of order 3 and degree 12.

In conclusion D_{14} is not a subgroup of a primitive group of class 12 of which the degree exceeds 20.

It is now an easy matter to show that G may not include D_{22}. The three substitutions of order 2 in D_{22} are s_1,

$$s_2 = a_1 a_3 \cdot b_1 b_3 \cdot c_1 c_3 \cdot d_1 d_3 \cdot e_1 e_3 \cdot f_1 f_2,$$
$$t \ = a_2 a_3 \cdot b_2 b_3 \cdot c_2 c_3 \cdot d_2 d_3 \cdot e_2 e_3 \cdot f_1 f_2.$$

Among the substitutions of G similar to s_1 there is a substitution s_3 which replaces f_1 by a letter of another set of D_{22}. There is not imposed upon this s_3 any other condition. Let

$$s_3 = (a_1 f_1) \ldots\ldots$$

First suppose $s_1 s_3 = s_3 s_1$. The constituent on the letters $a_1, \ldots\ldots$ of $\{s_2, s_3 = (a_1 f_1)(a_2 f_2) \ldots\ldots\}$ is non-regular of degree 6 or more. But now that D_{14} has been rejected, this is not possible. Hence $s_1 s_3 \neq s_3 s_1$, and likewise $s_2 s_3 \neq s_3 s_2$. Suppose that $s_3 = (a_1 f_1) \ldots\ldots$ fixes f_2. Then the letters a_2 and a_3, which form a cycle of t, are also fixed. The product $ts_3 = (a_1 f_1 f_2) \ldots\ldots$ requires that $\{t, s_3\}$ be of order 6. But a characteristic of the groups of order 6 in our list is that each generator displaces at least one letter from each cycle of the other. If s_3 displaces f_2, it displaces also a_1 and a_3. The set $a_1, f_1, f_2, \ldots\ldots$ of $\{s_1, s_3\}$ can not include the letter a_3. For that would put a non-regular

constituent of degree greater than four in $\{s_1, s_3\}$, and the presence of f_1 and f_2 in different cycles requires that the constituent on the letters $a_1, \ldots.$ be regular of degree 6 or 8. Then

$$s_3 = (a_1 f_1)(f_2 b_1)(b_2 -) \ldots..$$

To justify the cycle $(f_2 b_1)$: obviously the four sets b, c, d, e can be interchanged at will; and if b_2 were to follow f_2, $\{s_2, s_3\}$ would be non-regular in the set $a_1, \ldots..$ First suppose the set $a_1, \ldots.$ to consist of 6 letters; then

$$s_3 = (a_1 f_1)(f_2 b_1)(b_2 a_2) \ldots.,$$

and from $\{s_2, s_3\}$ the fourth cycle is $(b_3 a_3)$ or else

$$s_3 = (a_1 f_1)(f_2 b_1)(b_3 c_1)(c_3 a_3)(b_2 a_2) \ldots.$$

or

$$(a_1 f_1)(f_2 b_1)(b_3 c_3)(c_1 a_3)(b_2 a_2) \ldots.,$$

and in all three cases s_3 has a cycle entirely new to s_1, which D_{13} does not admit. Then suppose $\{s_1, s_3\}$ and $\{s_2, s_3\}$ both of order 8:

$$s_3 = (a_1 f_1)(f_2 b_1)(b_2 -) \ldots..$$

Now s_3 will clearly leave fixed the letters of one set of D_{22}, the letters e_1, e_2, e_3, say. Hence both $\{s_1, s_3\}$ and $\{s_2, s_3\}$ require that s_3 have the form of the "s_2" in D_{11}, that is, s_3 has no cycle new to s_1 or to s_2. Let us see if this is possible. A consequence is that c_1 and not c_3 follows b_2 in s_3. Then

$$s_3 = (a_1 f_1)(f_2 b_1)(b_2 c_1)(c_2 a_2) \ldots.,$$

which after all has a cycle new to s_2.

Thus D_{22} is no longer to be considered.

The group D_{12} is of order 8 and has but three transitive sets. The substitutions of order 2 and degree 12 in it are s_1,

$$\begin{aligned}
s_2 &= a_1 b_2 \cdot b_1 c_2 \cdot c_1 d_2 \cdot d_1 a_2 \cdot e_1 f_1 \cdot \alpha_1 \alpha_2, \\
t_1 &= a_1 d_2 \cdot b_1 a_2 \cdot c_1 b_2 \cdot d_1 c_2 \cdot e_2 f_2 \cdot \alpha_1 \alpha_2, \\
t_2 &= a_1 c_2 \cdot b_1 d_2 \cdot c_1 a_2 \cdot d_1 b_2 \cdot e_1 f_2 \cdot e_2 f_1, \\
t_3 &= a_1 c_1 \cdot b_1 d_1 \cdot a_2 c_2 \cdot b_2 d_2 \cdot e_1 f_1 \cdot e_2 f_2.
\end{aligned}$$

There is a substitution s_3, similar to s_1, which unites the set $a_1 a_2 b_1 b_2 c_1 c_2 d_1 d_2$ to one of the other sets of D_{12}, and which has not more than one new letter in any cycle. This s_3 is subject to the condition that it displaces as few letters as any substitution of the series $s_1, \ldots.$ which replaces a_1 by a letter of one of the other two sets. The degree of the extended constituent is 12, 16 or 18. If the degree is 18, H_3 is transitive. Let

$$s_3 = (a_1 e_1)(--)(\beta_1 -)(\beta_2 -)(\beta_3 -)(\beta_4 -).$$

Now s_3 is not commutative with any of the five substitutions of order 2 above; and since H_3 is transitive, displaces both α_1 and α_2. Therefore

$$s_3 = (a_1 e_1)(\alpha_1 —)(\alpha_2 \beta_1)(\beta_2 —)(\beta_3 —)(\beta_4 —)(a_2)(e_2)(b_2)(f_1)(c_2)(f_2)(c_1),$$

and but b_1, d_1 and d_2 are left for the remaining four places. Since D_{12} is invariant under all the substitutions of the group

$$\{b_1 d_1 \cdot b_2 d_2 \cdot e_1 e_2 \cdot f_1 f_2, \quad e_1 f_1 \cdot e_2 f_2 \cdot \alpha_1 \alpha_2\},$$

there remains in this connection only

$$s_3 = (a_1 \alpha_1)(— —)(\beta_1 —)(\beta_2 —)(\beta_3 —)(\beta_4 —).$$

Now s_3 fixes α_2, so that $\{s_1, s_3\}$ is of degree 17.

Suppose H_3 transitive of degree 16, and first let

$$s_3 = (a_1 e_1)(\beta_1 —)(\beta_2 —)(\alpha_1 —)(— —)(— —).$$

If $\{s_1, s_3\}$, $\{t_2, s_3\}$ and $\{t_3, s_3\}$ are of degree 15 (D_{13}), s_3 displaces a_2, e_2, c_2, f_2, c_1 and f_1 in its last two cycles, an absurdity. Hence s_3 displaces α_2, and in consequence fixes a_2, e_2, c_2, f_2, c_1, f_1. Now s_3 can not be formed for lack of letters. Let

$$s_3 = (a_1 \alpha_1)(\alpha_2 —)(\beta_1 —)(\beta_2 —) \ldots .$$

One of the letters e_1, e_2, f_1, f_2 is in the cycle with α_2. If necessary, transform s_3 by a substitution of

$$\{b_1 d_1 \cdot b_2 d_2 \cdot e_1 e_2 \cdot f_1 f_2, \quad e_1 f_1 \cdot e_2 f_2\},$$

so that s_3 may be written

$$(a_1 \alpha_1)(e_1 \alpha_2)(\beta_1 —)(\beta_2 —) \ldots .$$

Since $\{s_1, s_3\}$ and $\{t_4, s_3\}$ are of degree 16, they require the contradictory forms for s_3:

$$(\beta_1 a_2)(\beta_2 e_2) \ldots \quad \text{and} \quad (\beta_1 c_1)(\beta_2 f_1) \ldots .$$

Then H_3 is transitive.

Suppose the larger constituent of degree 16. Then

$$s_3 = (a_1 e_1)(\beta_1 —)(\beta_2 —)(\beta_3 —)(\beta_4 —)(— —),$$

where α_1 or α_2, if displaced, is in the last cycle only. Now s_3 fixes a_2, e_2, b_2, f_1, c_2, f_2, c_1, leaving only the five letters b_1, d_1, d_2, α_1, α_2 with the aid of a new letter γ to fill the six places vacant in s_3. But if γ is displaced, α_1 or α_2 is certainly fixed.

Suppose the first constituent to be of degree 12. Let s_3 unite the sets $a_1, \ldots$ and $e_1, \ldots$ of D_{12}. There is no new letter in this extended constituent, and s_3 can not displace more than two new letters. Assume then

$$s_3 = (a_1 e_1)(\alpha_1 \beta_1)(\alpha_2 \beta_2) \ldots .$$

The three groups $\{s_1, s_3\}$, $\{t_2, s_3\}$ and $\{t_3, s_3\}$ are Abelian. Hence

$$s_3 = (a_1 e_1)(a_2 e_2)(c_1 f_1)(c_2 f_2)(\alpha_1 \beta_1)(\alpha_2 \beta_2),$$

which is inconsistent with s_2. Let

$$s_3 = (a_1 e_1)(\alpha_1 \beta) \ldots,$$

leaving α_2 fixed. One constituent of H_3 is of order 6, and to identity corresponds an invariant subgroup of degree 12 and class 12: the group $\{s_1, t_2\}$. Hence the larger constituent is of order 24. Its class is 10. Its positive subgroup has four subgroups of order 3 and hence is the regular tetrahedral group. Hence s_3 is not commutative with t_3, but is commutative with s_1 or else transforms s_1 into t_3. First assume $s_3 s_1 = s_1 s_3$, and let

$$s_3 = a_1 e_1 \cdot a_2 e_2 \cdot \alpha_1 \beta \ldots (\alpha_2) \ldots .$$

Since $(a_2 \alpha_1)$ is not a cycle of s_2, $\{s_2, s_3\}$ is not of degree 14, and s_3 displaces f_2:

$$s_3 = (a_1 e_1)(b_2 -)(f_1 -)(a_2 e_2)(\alpha_1 \beta)(f_2 -)(\alpha_2)(d_1).$$

Now that cycle which s_3 has in common with s_1 is evidently $(b_1 b_2)$, and since $\{t_1, s_3\}$ is of degree 15, d_2 is fixed. Then $(c_1 f_2)$ is a cycle of s_3. Therefore when $s_1 s_3 = s_3 s_1$, s_3 is determined as

$$a_1 e_1 \cdot b_1 b_2 \cdot f_1 c_2 \cdot a_2 e_2 \cdot \alpha_1 \beta \cdot c_1 f_2 .$$

The substitution $a_2 c_2 \cdot b_2 d_2 \cdot e_1 f_2 f_1 e_2$ transforms s_1 into t_2, s_2 into t_1 and $s_1 s_3$ into a substitution $s_3' = a_1 e_1 \cdot c_2 f_2 \cdot \alpha_1 \beta \ldots$ commutative with t_2 and not with s_1. Hence s_3 as determined above is unique.

We take with H_3 a substitution s_4, which with it generates a transitive group on 16, 18 or 20 letters. If the degree is 16 or 18, systems of imprimitivity of 8 or 9 letters are impossible. Nor, because of

$$s_2 s_3 = a_1 f_1 b_1 \cdot a_2 e_2 d_1 \cdot b_2 c_2 e_1 \cdot c_1 f_2 d_2 \cdot \alpha_1 \beta \alpha_2,$$

are systems of two letters possible. There may not be more than one system containing a given letter (γ) of three, four or six letters. Then we have only to study the transitive group of degree 20. Let

$$s_4 = (a_1 \alpha_1)(\gamma_1 -)(\gamma_2 -)(\gamma_3 -)(\gamma_4 -)(\gamma_5 -).$$

At once we note that s_4 fixes b_2, α_2, d_2, e_1, β, and from $s_1 s_3$, e_2. Then $\{s_1, s_4\}$ is D_{18}, so that

$$s_4 = a_1 \alpha_1 \cdot a_2 \gamma_1 \cdot c_1 \gamma_2 \cdot c_2 \gamma_3 \cdot d_1 \gamma_4 \cdot d_2 \gamma_5,$$

but $\{t_3, s_4\}$ is also D_{18}, now impossible.

Now H_3 can not be a group with one constituent of degree 12 and one of degree 2. For the subgroup leaving fixed the two letters of the second set is of class 12 and can only be of order 4.

If s_3 joins the sets $a_1, \ldots.$ and $\alpha_1, \ldots.$ of D_{12} (to form a constituent of degree 12 in H_3), there are just two new letters in this set, while the degree of the other constituent is 4 or 6. Let

$$s_3 = (a_1 \alpha_1)(\beta_1 -)(\beta_2 -)(- -)(\gamma_1 -)(\gamma_2 -).$$

Clearly s_3 fixes α_2 and $\{s_1, s_3\}$ is of degree 17. Let

$$s_3 = (a_1 \alpha_1)(\beta_1 -)(\beta_2 -) \ldots.,$$

with no new letters other than β_1 and β_2. In case this substitution s_3 can be set up without contradiction, there is a substitution $s_4 = (a_1 e_1) \ldots.$ with at most one new letter in any cycle; and if

$$s_4 = (a_1 e_1)(\beta_1 \delta_1) \ldots.,$$

for example, consider the transform of D_{12} by s_4, bearing in mind that no substitution of the series $s_4, \ldots.$ that connects the two sets of H_3 displaces fewer new letters than s_4. The transform of D_{12} by s_4 fixes β_1 and δ_1. Then with the substitutions s_1, s_2, t_1, t_2, t_3 before us it is seen that s_4 must replace the letters a_2, b_2, d_2, c_2, c_1 by letters not a part of the set $a_1, a_2, \ldots., d_2$; that is, that these five letters must be in different cycles of s_4, which is impossible. Hence s_4 does not replace β_1 or β_2 by a new letter, but is simply a substitution uniting $a_1, \ldots.$ and $e_1, \ldots.$ in a set of degree 12 or more, and with at most one letter new to D_{12} in any cycle. It has already been seen that this is not possible.

With D_{12}, D_{11} disappears from our list.

Consider D_{13}. We shall make use of the substitutions s_1,

$$s_2 = a_1 b_1 \cdot a_2 c_1 \cdot b_2 c_2 \cdot d_1 \delta \cdot e_1 \varepsilon \cdot f_1 \zeta, \quad t = a_1 c_2 \cdot a_2 b_2 \,.\, b_1 c_1 \cdot d_2 \delta \cdot e_2 \varepsilon \cdot f_2 \zeta,$$
$$s_1 s_2 = a_1 c_1 b_2 \cdot a_2 b_1 c_2 \cdot d_1 d_2 \delta \cdot e_1 e_2 \varepsilon \,.\, f_1 f_2 \zeta.$$

Since there is in D_{13} a set of six letters, there is a substitution s_3 similar to s_1 which replaces a_1 by a letter of some of the three other sets and which has at most one new letter in any cycle. The substitutions of the group

$$\{d_1 e_1 \cdot d_2 e_2 \cdot \delta \varepsilon, \quad e_1 f_1 \cdot e_2 f_2 \cdot \varepsilon \zeta, \quad b_1 c_2 \cdot b_2 c_1 \cdot d_1 d_2 \cdot e_1 e_2 \cdot f_1 f_2,$$
$$a_2 b_1 \cdot b_2 c_1 \cdot d_2 \delta \cdot e_2 \varepsilon \cdot f_2 \zeta\}$$

transform D_{13} into itself and hence we need only put

$$s_3 = (a_1 d_1) \ldots..$$

If $\{s_1, s_2, \ldots.\}$ is a transitive group on 16 or 18 letters, systems of imprimitivity of two, eight or nine letters are not possible. A system of three letters

can be chosen in but one way, and a system of four or six letters in three ways at most. Then D_{18} can not lead to a transitive group of lower degree than 20. Let

$$s_3 = (a_1 d_1)(\gamma_1 -)(\gamma_2 -)(\gamma_3 -)(\gamma_4 -)(\gamma_5 -),$$

without condition upon the number of constituents in H_3. There is no cycle of new letters in s_3. Since $\{s_1, s_3\}$ and $\{s_2, s_3\}$ are non-Abelian, s_3 fixes a_2, d_2, b_1 and ζ, and again from s_1, b_2. Both a_2 and b_2 being fixed, $\{t, s_3\}$ is D_{18}, so that

$$s_3 = a_1 d_1 \cdot c_2 \gamma_1 \cdot d_2 \gamma_2 \cdot \delta \gamma_3 \cdot e_2 \gamma_4 \cdot \varepsilon \gamma_5,$$

which, however, generates with s_1 a group of degree 17.

Let us remove the condition that there is no cycle of new letters in s_3. Suppose γ_5 replaced by a letter β. The group $\{s_1, s_3\}$ is of degree 18 and non-Abelian. Then s_3 fixes a_2, d_2, b_1, ζ, b_2, ε and δ, so that $\{t, s_3\}$ is of degree greater than 18, an impossibility. Nor could there be more cycles new to D_{18} in s_3. In the same way let

$$s_3 = (a_1 d_1)(--)(\gamma_1 -)(\gamma_2 -)(\gamma_3 -)(\gamma_4 -),$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are the only letters new to D_{18} in s_3. Since $\{s_1, s_3\}$ and $\{s_2, s_3\}$ are non-Abelian, s_3 fixes a_2, d_2, b_1 and δ. The group $\{t, s_3\}$ is of degree 18 (d_2 and δ being fixed by s_3), so that s_3 displaces e_1 or f_1, not both, and replaces c_2 by a letter new to t. If s_3 replaces c_2 by e_1 or f_1, s_4 fixes c_1, b_2, e_2 or f_2, and ε or ζ. Not enough letters are left to replace the four $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. Hence

$$s_3 = (a_1 d_1)(c_2 \gamma_2) \ldots\ldots$$

From s_1, c_1 is replaced by a letter new to s_1, and from t, b_1 being fixed, c_1 is replaced by ε or ζ, not by $\gamma_1, \ldots\ldots$ Then s_3 fixes e_2 or f_2, and since $\{s_2, s_3\}$ can not be of degree 17, s_4 fixes both e_2 and f_2. But from $\{s_1, s_3\}$, if f_1 is replaced by a γ, so also is f_2. Since now e_1, e_2, f_1, f_2 and b_1 are fixed, $\{s_1, s_3\}$ is impossible. Suppose s_3 has one other new letter (β) in the cycle ($\beta \gamma_4$). If $s_3 s_1 = s_1 s_3$,

$$s_3 = a_1 d_1 \cdot a_1 d_2 \cdot \delta \gamma_1 \cdot \varepsilon \gamma_2 \cdot \zeta \gamma_3 \cdot \beta \gamma_4,$$

which is not consistent with s_2. Then $\{s_1, s_3\}$ is of degree 18, and s_3 fixes a_2 and d_2. From s_2, s_3 fixes b_1 and δ; from t, it fixes e_1 and f_1. Now from s_1, s_3 displaces one, not both, the letters ε and ζ, while from s_2, either $s_3 = (\varepsilon \zeta) \ldots\ldots$ or fixes both ε and ζ. If s_3 has two or more letters $\beta_1, \beta_2, \ldots\ldots$ in the last four cycles, $\{t, s_3\}$ is impossible.

We now are in position to say that if s is a substitution of G similar to s_1, which replaces a_1 by a letter of one of the three remaining sets of D_{18}, s has new letters in not more than three of its cycles.

Let s_3 displace just three new letters in three cycles:

$$s_3 = (a_1 d_1)(--)(--)(\gamma_1 -)(\gamma_2 -)(\gamma_3 -).$$

If s_1 and s_3 are commutative,

$$s_3 = (a_1 d_1)(a_2 d_2)(--)(\delta \gamma_1)(\varepsilon \gamma_2)(\zeta \gamma_3),$$

but since $\{s_2, s_3\}$ is of degree 16 or more, s_3 fixes δ. Then s_3 is not commutative with s_1, nor, in like manner, with s_2. If $\{s_1, s_3\}$ is of degree 15,

$$s_3 = (a_1 d_1)(a_2 -)(d_2 -)(\gamma_1 -)(\gamma_2 -)(\gamma_3 -),$$

and fixes, since $\{s_2, s_3\}$ is of higher degree, b_1 in addition to δ, ε, and ζ. If c_2 is fixed, the second and third cycles of s_3, because of t, contain e_1 and f_1, which s_1 does not admit. From t, c_2 follows γ_1; from s_2 and s_1, b_2 is replaced by a letter new to s_2 and to s_1, by γ_2; a_2 is replaced by a letter of s_2 (c_1 being fixed) that is new to t, by e_1:

$$s_3 = (a_1 d_1)(a_2 e_1)(d_2 e_2)(\gamma_1 c_2)(\gamma_2 b_2)(\gamma_3 -).$$

From s_2 the remaining letter should be f_2, and from t, f_1. After transforming $\{D_{13}, (a_1 d_1) \ldots .\}$ by $a_2 b_1 \cdot b_2 c_1 \cdot d_2 \delta \cdot e_2 \varepsilon \cdot f_2 \zeta$, we conclude that both $\{s_1, s_3\}$ and $\{s_2, s_3\}$ are of degree 16 or more. The letters a_2, d_2, b_1, δ, ε or ζ, e_2 or f_2 are fixed by s_3. As there remain seven places to fill in s_3, no other letter of D_{13} is fixed by s_3. Now c_2 must be replaced by a letter new to t, by γ_1 or by e_1. If $s_3 = (c_2 e_1) \ldots .$, c_1 is fixed. Now if $s_3 = (a_1 d_1)(c_2 \gamma_1) \ldots .$, c_1 is followed by a letter of t, not in s_1: by ε or ζ, whence e_1 or f_1 is fixed.

It is proved that s_3 has not just three new letters, one to a cycle.

Let s_3 have just two new letters in distinct cycles. If s_3 and s_1 are commutative we can not have

$$s_3 = (a_1 d_1)(a_2 d_2)(\gamma_1 \delta)(\gamma_2 -) \ldots .,$$

because then the set $a_1, \ldots .$ of $\{s_2, s_3\}$ is non-regular of degree 5 or more. Then put

$$s_3 = (a_1 d_1)(a_2 d_2)(\gamma_1 \varepsilon)(\gamma_2 \zeta) \ldots . (\delta).$$

Both $\{s_2, s_3\}$ and $\{t, s_3\}$ are of degree 16, so that e_1, f_1, e_2, f_2 are replaced by letters new to s_2 and t. But this makes $\{s_2, s_3\}$ of degree 17. Then s_3 is not commutative with s_1 or s_2. Suppose both $\{s_1, s_3\}$ and $\{s_2, s_3\}$ of degree 16. Now s_3 fixes a_2, d_2, b_1, δ, and e_1, say, and can fix no other letter of D_{13}. From s_1, b_2 must be replaced by e_2, which, since $\{t, s_3\}$ is also of degree 16, is not possible. Suppose $\{s_1, s_3\}$ of degree 15 and $\{s_2, s_3\}$ of degree 16: s_3 fixes

b_1, δ, e_2, say, ε or ζ, e_1 or f_1. No other letter of D_{13} is fixed by s_3. The group $\{t, s_3\}$ is of degree 16.

$$s_3 = (a_1 d_1)(a_2 -)(d_2 -)(- -)(\gamma_1 -)(\gamma_2 -).$$

From t, c_1 is followed by d_2 or e_1. But

$$s_3 = (a_1 d_1)(a_2 c_2)(d_2 c_1) \ldots .$$

requires that s_3 fixes c_1 and b_2, and

$$s_3 = (a_1 d_1)(a_2 -)(d_2 -)(c_1 e_2)(\gamma_1 -)(\gamma_2 -)$$

has but two letters new to s_1. Then both $\{s_1, s_3\}$ and $\{s_2, s_3\}$ are of degree 15; s_3 fixes two of the letters δ, ε, ζ, both e_2 and f_2, and only one of the letters e_1 and f_1, e_1, say. Hence $\{s_1, s_3\}$ is not of degree 15 (e_1 and e_2 being fixed).

Let H_3 be intransitive and of degree 16. The constituent $a_1, \ldots, d_1, \ldots$ is of degree 9 or 12. Then the letter γ is in the set f_1, f_2, ζ on four letters. Three constituents of the degrees 9, 3, and 4 entail the presence of a substitution of degree 12 and order 3 in H_3. Hence H_3 has just two constituents. Now s_4 must have letters new to H_3 in at least four cycles, and can therefore only be

$$s_4 = (a_1 \gamma_1) \ldots .$$

Unless s_3 replaces a_1, a_2, b_1, b_2, c_1, c_2 by letters not belonging to this set of six letters, it is possible to choose a substitution σ from D_{13} such that $s_3 \sigma^{-1} s_4 \sigma s_3$ replaces a letter of the set $a_1, \ldots$ of D_{13} by a letter of another set (f_1, f_2, ζ) and has letters new to D_{13} in more than three cycles. The condition that s_3 fixes a letter $a_1, \ldots, c_2$ or transposes two of them is satisfied, because s_3 uses one cycle to transpose an $f_1, \ldots$ and γ.

If H_3 is of degree 15, s_4 certainly displaces letters new to D_{13}, without a cycle of new letters. But we have shown that s_4 does not exist.

Any non-Abelian diedral rotation group generated by two substitutions similar to s_1, and contained in a primitive group of class 12 and of degree greater than 20, is therefore of degree 16 or 18.

The group which now seems to offer least difficulty is D_1. Other than the identity its substitutions are s_1,

$$s_2 = a_1 b_1 \cdot a_2 b_2 \cdot c_1 d_1 \cdot c_2 d_2 \cdot e_1 f_1 \cdot e_2 f_2,$$
$$t = a_1 b_2 \cdot a_2 b_1 \cdot c_1 d_2 \cdot c_2 d_1 \cdot e_1 f_2 \cdot e_2 f_1.$$

All substitutions similar to s_1, which unite sets of D_1, may be transformed into $s_3 = (a_1 c_1) \ldots$ by

$\{a_1 c_1 \cdot a_2 c_2 \cdot b_1 d_1 \cdot b_2 d_2,\ a_1 e_1 \cdot a_2 e_2 \cdot b_1 f_1 \cdot b_2 f_2,\ a_2 b_1 \cdot c_2 d_1 \cdot e_2 f_1,\ b_1 b_2 \cdot d_1 d_2 \cdot f_1 f_2\}$, under which D_1 is invariant.

First suppose that no s_3 which joins sets of D_1 is free from a cycle of new letters. Then s_3 displaces all the letters of the sets united. Since H_3 can not be regular of order 12, all three sets are not united. Now s_3 displaces the eight letters $a_1, \ldots, c_1, \ldots,$ and is therefore commutative with s_1 and s_2. Hence it fixes each of the four letters e_1, e_2, f_1, f_2, and is uniquely

$$s_3 = a_1 c_1 \cdot a_2 c_2 \cdot b_1 d_1 \cdot b_2 d_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2.$$

This s_3 is obtained under the assumption that it displaces the minimum number of new letters. For later developments consider any substitution

$$s = (a_1 c_1)(\alpha_1 \alpha_2) \ldots .$$

of degree 12. If s is commutative with both s_1 and s_2, it is identical with s_3 above. Let $s_1 s_2 \neq s_2 s_1$. Then s has at least four letters new to D_1, and fixes a_2 and c_2. The substitution $s s_1 s$ has at most, and hence exactly, four letters new to s_1, and since it joins sets of D_1, has a cycle of new letters; but any substitution, as $s s_1 s$, that joins sets of D_1, displaces the minimum number of new letters, and has a cycle of new letters, is of the same type as s_3. Then

$$s s_1 s = a_2 c_1 \cdot a_1 c_2 \cdot b_1 d_2 \cdot b_2 d_1 \cdot \gamma_1 \gamma_2 \cdot \gamma_3 \gamma_4,$$

so that

$$s = a_1 c_1 \cdot \alpha_1 \alpha_2 \cdot e_1 \gamma_1 \cdot e_2 \gamma_2 \cdot f_1 \gamma_3 \cdot f_2 \gamma_4.$$

Then

$$s s_2 s = c_1 b_1 \cdot a_2 b_2 \cdot a_1 d_1 \cdot c_2 d_2 \cdot \gamma_1 \gamma_3 \cdot \gamma_2 \gamma_4,$$

not, as required, of the form of s_3. Hence any substitution $(a_1 c_1)(\alpha_1 \alpha_2) \ldots .$ of degree 12 coincides with s_3.

Let us next study a substitution

$$s = (a_1 \alpha) \ldots .$$

of degree 12. This substitution $(a_1 \alpha) \ldots .$ can join no sets of D_1. The number of new letters displaced by it is 6, one to a cycle. One and only one of the groups $\{s_1, s\}$, $\{s_2, s\}$, $\{t, s\}$ is of order 8 (D_{18}).

We now seek the substitution s_4, $(a_1 e_1) \ldots .$ or $(a_1 \alpha_1) \ldots .$, with at most one new letter in any cycle. Every substitution similar to s_1, which joins sets of such a group as D_1, has, like s_3, four new letters. Now $\{D_1, s_4\}$ is such a group, and hence the first four cycles of s_4 are fixed as $a_1 e_1 \cdot a_2 e_2 \cdot b_1 f_1 \cdot b_2 f_2$. The product $s_2 s_4$ is of order 3, so that

$$s_4 = a_1 e_1 \cdot a_2 e_2 \cdot b_1 f_1 \cdot b_2 f_2 \cdot \alpha_1 \alpha_3 \cdot \beta_1 \beta_3.$$

Before considering s_5, let $s_4 = (a_1 \alpha_1) \ldots .$ Suppose $\{s_1, s_4\}$ of order 8. If $s_3 s_4 = s_4 s_3$,

$$s_4 = a_1 \alpha_1 \cdot c_1 \alpha_2 \cdot a_2 \beta_1 \cdot c_2 \beta_2 \cdot e_1 \gamma_1 \cdot e_2 \gamma_2,$$

after having transformed H_4 by $e_1 f_1 \cdot e_2 f_2$, if necessary. If $s_3 s_4 \neq s_4 s_3$, s_4 fixes c_1, α_2, b_1, b_2, c_2, both e's or else both f's. Hence

$$s_4 = (a_1 \alpha_1)(a_2 -)(d_1 -)(d_2 -)(e_1 -)(e_2 -).$$

Now $\{s_3, s_4\}$ must be of degree 18, and hence a_2, d_1 or d_2 is replaced by a letter new to s_3, requiring s_4 to displace c_2, b_1 or b_2. The substitution $b_1 b_2 a_2 \cdot d_1 d_2 c_2 f_1 f_2 e_2$ permutes s_1, s_2, t cyclically, transforms s_3 into itself, and fixes the cycle $(a_1 \alpha_1)$ of s_4. Hence it is not necessary to investigate $(a_1 \alpha_1) \ldots .$ under the assumption that $\{s_2, s_4\}$ is of order 8.

There are two forms s_4 may have:

$$s_4' = a_1 \alpha_1 \cdot c_1 \alpha_2 \cdot a_2 \beta_1 \cdot c_2 \beta_2 \cdot e_1 \gamma_1 \cdot e_2 \gamma_2,$$
$$s_4'' = a_1 e_1 \cdot a_2 e_2 \cdot b_1 f_1 . b_2 f_2 \cdot \alpha_1 \alpha_3 \cdot \beta_1 \beta_3.$$

The group $\{H_3, s_4'\} \equiv H_4'$ has but two constituents. If $s_5 = (a_1 e_1) \ldots .,$ from $\{s_4', s_1 s_4' s_1\}$ and $\{s_1, s_2\}$, at once

$$s_5 = a_1 e_1 \cdot a_2 e_2 \cdot b_1 f_1 \cdot b_2 f_2 \cdot \alpha_1 \gamma_1 \cdot \beta_1 \gamma_2.$$

We now have before us a transitive group of degree 18, in which no system of imprimitivity of q letters involving γ_1 can be chosen in $q+1$ ways. Hence we are not concerned with this group. If $s_5 = (a_1 f_1) \ldots .,$ from D_1,

$$s_5 = a_1 f_1 \cdot a_2 f_2 \cdot b_1 e_1 \cdot b_2 e_2 \ldots .,$$

and if $s_5 = (a_1 \gamma_1) \ldots .,$ from $\{s_4', s_1 s_4' s_1\}$,

$$s_5 = a_1 \gamma_1 \cdot e_1 \alpha_1 \cdot a_2 \gamma_2 \cdot e_2 \beta_2 \ldots . ;$$

then $\{H_4', s_5\}$ is transitive of degree 20 at most. Hence G is more than simply transitive. We may transform s_1 into $(a_1 e_1) \ldots .$ and this substitution must coincide with that form of s_5 above which gave the transitive group of degree 18. Having thrown out s_4', we may remark that H_3 can not be a subgroup of a doubly transitive group. For were that the case s_1 could be transformed into $(a_1 \alpha_1) \ldots . .$ In particular $\{s_1, s_2, s_3, \ldots .\}$ must not lead us to a transitive group of degree less than 21. Consider

$$s_4'' = a_1 e_1 \cdot a_2 e_2 \cdot b_1 \cdot f_1 \cdot b_2 f_2 \cdot \alpha_1 \alpha_3 \cdot \beta_1 \beta_3,$$

and $s_5 = (a_1 \alpha_3) \ldots . .$ Now s_5 bears the same relation to $\{s_1, s_2, s_4''\}$ as s_4' does to $\{s_1, s_2, s_3\}$, so that $s_5 s_4'' = s_4'' s_5$, and s_5 replaces the letters of the last two cycles of s_4'' by the letters of two of the preceding cycles. Then $\{s_1, s_2, s_3, s_4'', s_5\}$ is transitive of degree less than 21.

Finally assume that at least one substitution s_3 is to be found in the series

$s_1, s_2, \ldots.$ which connects two or more sets of D_1 and has no cycle of new letters. Let

$$s_3 = (a_1 c_1) \ldots..$$

s_3 is not commutative with s_1, s_2 or t. Hence s_3 fixes the six remaining letters of the sets $a_1, \ldots.$ and $c_1, \ldots.$ of D_1, requiring six new letters, while five is the most it can displace.

Since D_1 is a subgroup of D_{18} and D_{19}, these two groups are, with D_1, to be struck from our list.

Now consider D_2. The substitutions of order 2 are s_1,

$$s_2 = a_1 b_1 \cdot a_2 b_2 \cdot c_1 d_1 \cdot c_2 d_2 \cdot e_1 e_2 \cdot \alpha_1 \alpha_2, \quad t = a_1 b_2 \cdot a_2 b_1 \cdot c_1 d_2 \cdot c_2 d_1 \cdot f_1 f_2 \cdot \alpha_1 \alpha_2.$$

Since D_2 is invariant under all the substitutions of the group

$$\{c_1c_2 \cdot d_1d_2,\ c_1d_1 \cdot c_2d_2,\ e_1e_2,\ f_1f_2,\ \alpha_1\alpha_2,\ b_1b_2 \cdot d_1d_2 \cdot e_1f_1 \cdot e_2f_2,\ a_2b_1 \cdot c_2d_1 . f_1\alpha_1 \cdot f_2\alpha_2\},$$

we reduce all substitutions which replace a_1 by a letter of one of the other sets to $(a_1 c_1) \ldots.$ or to $(a_1 e_1) \ldots..$ Without condition upon the number of new letters involved, let

$$s_3 = (a_1 e_1) \ldots..$$

Obviously it is not commutative with either s_1 or s_2. Then it fixes a_2, b_1 or e_2, and replaces b_2 by a letter new to t, but in s_1 and s_2, which is absurd. Hence:

No substitution of G, similar to s_1, transposes one of the letters a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 with e_1, e_2, f_1, f_2, α_1 or α_2.

Now assume that s_3 displaces the minimum number of letter new to D_2, and that every such substitution has a cycle of new letters. Then s_3 displaces all the letters of the set $a_1 \ldots.$ of H_3, and in consequence is commutative with both s_1 and s_2. Hence

$$s_3' = a_1 c_1 \cdot a_2 c_2 \cdot b_1 d_1 \cdot b_2 d_2 \cdot e_1 e_2 \cdot \gamma_1 \gamma_2$$

or

$$s_3 = a_1 c_1 \cdot a_2 c_2 \cdot b_1 d_1 \cdot b_2 d_2 \cdot \gamma_1 \gamma_2 \cdot \delta_1 \delta_2 .$$

The next substitution $s_4 = (a_1 \gamma_1) \ldots.$ has no cycle of letters new to H_3.

Let us first attend to s_3'. Since $\{s_1, s_3'\}$ is a group D_2, the substitution $s_4 = (a_1 \gamma_1) \ldots.$ is not possible.

Consider s_3, with the assumption that $s_3 s_4 = s_4 s_3$. Then

$$s_4 = (a_1 \gamma_1)(c_1 \gamma_2) \ldots..$$

If s_4 displaces a_2, from s_1, a_2 and c_2 are replaced by letters new to s_1, so that s_4 interchanges the cycles $(a_2 c_2)$ and $(\delta_1 \delta_2)$ of s_3. Now $s_2 s_4$ and $t s_4$ can only be of order 3; hence

$$s_4 = a_1 \gamma_1 \cdot c_1 \gamma_2 \cdot a_2 \delta_1 \cdot c_2 \delta_2 \cdot e_1 f_1 \cdot \alpha_1 \alpha_3 .$$

Suppose a_2 fixed. If b_1 (or b_2) is displaced we can transform the resulting substitution into the s_4 just given. Let s_4 fix a_2, b_1 and b_2. Then $\{s_1, s_4\}$ is impossible. Next assume that s_3 and s_4 are not commutative. The letters c_1 and γ_2 are fixed. Suppose a_2 displaced by s_4. It is followed by a letter new to s_1: by α_1, δ_1 or ε. The first

$$(a_1 \gamma_1)(a_2 \alpha_1)\ldots.$$

is impossible because from s_2, b_1 is displaced by s_4, while from t, it is seen to be fixed. Let

$$s_4 = (a_1 \gamma_1)(a_2 \delta_1)\ldots..$$

The letters c_1, c_2, γ_2, δ_2 are fixed. If s_4 and s_3 have a cycle in common it is $(b_1 d_1)$ or $(b_2 d_2)$. In either event $\{s_2, s_4\}$ is rendered impossible. Now since $\{s_3, s_4\}$ is D_{15} or D_{21}, and since $b_1 d_1 \cdot b_2 d_2 \cdot e_1 f_1 \cdot e_2 f_2$ transforms H_3 into itself,

$$s_4 = (a_1 \gamma_1)(a_2 \delta_1)(b_1 -)(d_1 -)\ldots.,$$

where b_1 and d_1 are followed by letters not merely new to s_3, but also new to D_2. On consulting s_1, this is seen to be impossible. If

$$s_4 = (a_1 \gamma_1)(a_2 \varepsilon)\ldots.(c_1)\ldots.,$$

c_2 is replaced by a letter of s_1 new to s_3, an impossibility. Assume that s_4 fixes a_2. We may add that it fixes b_1 and b_2. Now $\{s_1, s_4\}$ is impossible.

The unique substitution

$$s_4 = a_1 \gamma_1 \cdot c_1 \gamma_2 \cdot a_2 \delta_1 \cdot c_2 \delta_2 \cdot e_1 f_1 \cdot \alpha_1 \alpha_3$$

generates with H_3 a group with three transitive constituents. We need consider only

$$s_5 = (a_1 \alpha_3)\ldots.;$$

and since

$$s_1 s_5 s_1 = (b_1 \alpha_3)\ldots.,$$

and

$$s_4 s_1 s_5 s_1 s_4 = (b_1 \alpha_1)\ldots.,$$

this case is complete.

There remains the hypothesis that a substitution s_3 exists without a cycle of new letters. Let s_3 be commutative with s_1. It is not commutative with s_2. Then b_1, b_2, d_1, d_2 are fixed. If s_2 and s_3 have the cycle $(\alpha_1 \alpha_2)$ in common, s_4 has a cycle of letters new to D_2. If s_2 and s_4 have $(e_1 e_2)$ in common,

$$s_3' = a_1 c_1 \cdot a_2 c_2 \cdot e_1 e_2 \cdot f_1 f_2 \cdot \alpha_1 \beta_1 \cdot \alpha_2 \beta_2.$$

If s_3 has α_1 and α_2 in different cycles, and no cycle in common with s_2,

$$s_3'' = a_1 c_1 \cdot a_2 c_2 \cdot \alpha_1 \beta_1 \cdot \alpha_2 \beta_2 \cdot e_1 f_1 \cdot e_2 f_2.$$

Let s_3 be non-commutative with s_1. Since

$$b_1 b_2 a_2 \cdot d_1 d_2 c_2 \cdot e_1 \alpha_1 f_1 \cdot e_2 \alpha_2 f_2$$

permutes s_1, s_2 and t cyclically, we assume s_3 non-commutative with s_2 and t. Then s_3 fixes a_2, c_2, b_1, b_2, d_1, d_2, and must displace e_1, e_2, f_1, f_2, α_1 and α_2, an impossibility.

The next step is to set up a substitution s_4 with s_3' or s_3''. If we put $s_4 = (a_1 \beta_1) \ldots.$, $s_3 s_4 s_3 = (c_1 \alpha_1) \ldots..$

Then D_2 does not occur in G, and with it go D_{16}, D_{17}, D_{20} and D_{21}, in which D_2 is a subgroup.

Any two substitutions of G, similar to s_1, which have a common cycle, or such that one has a cycle new to the other, are commutative.

The next group we shall study is D_{15}. The substitutions of order 2 are

$$s_1 = a_1 a_2 \cdot b_1 b_2 \cdot c_1 c_2 \cdot d_1 d_2 \cdot e_1 e_2 \cdot f_1 f_2, \quad s_2 = a_1 b_1 \cdot c_1 d_1 \cdot e_1 \varepsilon_1 \cdot e_2 \varepsilon_2 \cdot f_1 \zeta_1 \cdot f_2 \zeta_2,$$
$$t_1 = a_2 b_2 \cdot c_2 d_2 \cdot e_2 \varepsilon_1 \cdot e_1 \varepsilon_2 \cdot f_2 \zeta_1 \cdot f_1 \zeta_2, \quad t_2 = a_1 b_2 \cdot a_2 b_1 \cdot c_1 d_2 \cdot c_2 d_1 \cdot \varepsilon_1 \varepsilon_2 \cdot \zeta_1 \zeta_2.$$

D_{15} is invariant under the substitutions of the group

$$\{c_1 d_1 \cdot c_2 d_2, \quad e_1 e_2 \cdot \varepsilon_1 \varepsilon_2 \cdot f_1 f_2 \cdot \zeta_1 \zeta_2, \quad a_2 b_2 \cdot c_2 d_2 \cdot e_1 \varepsilon_1 \cdot e_2 \varepsilon_2 \cdot f_1 \zeta_1 \cdot f_2 \zeta_2,$$
$$a_1 c_1 \cdot a_2 c_2 \cdot b_1 d_1 \cdot b_2 d_2, \quad e_1 f_1 \cdot e_2 f_2 \cdot \varepsilon_1 \zeta_1 \cdot \varepsilon_2 \zeta_2,$$
$$a_1 e_1 \cdot a_2 \varepsilon_1 \cdot b_1 e_2 \cdot b_2 \varepsilon_2 \cdot c_1 f_1 \cdot c_2 \zeta_1 \cdot d_1 f_2 \cdot d_2 \zeta_2\}.$$

The substitutions s_3 to be considered are

$$s_3' = (a_1 c_1) \ldots., \quad s_3'' = (a_1 e_1) \ldots., \quad s_3''' = (a_1 c_2) \ldots..$$

Assume that any substitution similar to s_1 which connects two or more sets of D_{15} has a cycle of new letters. Now $s_3'' = (a_1 e_1) \ldots.$ and $s_3''' = (a_1 c_2) \ldots.$ can not be commutative with both s_1 and s_2. Then we have uniquely

$$s_3 = a_1 c_1 \cdot a_2 c_2 \cdot b_1 d_1 \cdot b_2 d_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2.$$

Now s_4 must be $(a_1 \alpha_1) \ldots..$ This s_4 can have no cycle new to D_{15} and hence does not unite two sets of D_{15}. Suppose s_4 not commutative with s_3. Since now s_4 fixes c_1, it must displace d_1 and d_2, whence $\{s_1, s_4\}$ is of degree 16. Now s_4 should replace a_2 by a letter new to D_{15}, but this may not be, because s_4 and t_1 are commutative. Let s_4 and s_3 be commutative. If s_4 displaces a_2, it replaces a_2 by b_2, which $\{s_1, s_4\}$ does not permit. Then s_4 fixes a_2 and b_2. Both b_1 and b_2 can not be fixed by s_4, hence

$$s_4 = a_1 \alpha_1 \cdot c_1 \alpha_2 \cdot b_1 \beta_1 \cdot d_1 \beta_2 \cdot e_1 \varepsilon_2 \cdot f_1 \zeta_2.$$

We determined e_1 and f_1 by means of the transformations $e_1 e_2 \cdot \varepsilon_1 \varepsilon_2$ and $f_1 f_2 \cdot \zeta_1 \zeta_2$.

The substitution s_5 is now obviously out of the question.

There is in the series $s_1, \ldots.$ at least one substitution which connects sets of D_{15} and has no cycle of new letters. Let

$$s_3 = (a_1 c_1) \ldots..$$

Since $s_3 t_1 = t_1 s_3$, if s_3 is commutative with s_1 or t_2, it is commutative with all the substitutions of D_{15}. Then it is not commutative with s_1 or t_2. Suppose it is commutative with s_2. It fixes a_2, b_2, c_2, d_2, and displaces the same letters as s_2. Then s_3 fixes a_2 and b_1, leading to the absurd conclusion that it is commutative with t_2. Let

$$s_3 = (a_1 e_1) \ldots..$$

If s_3 is commutative with s_1, it is not commutative with s_2, and vice-versa. Suppose

$$s_3 = (a_1 e_1)(a_2 e_2) \ldots.,$$

fixing b_1, b_2, ε_1. Since $t_2 s_3$ is of order 3, one cycle of s_3 is $(\varepsilon_2 \gamma_1)$, and another (we may transform by $f_1 f_2 \cdot \zeta_1 \zeta_2$) is $(\zeta_2 \gamma_2)$. Then we also have $(f_2 c_2)$ or $(f_2 d_2)$, determined arbitrarily by means of $c_1 d_1 \cdot c_2 d_2$ as $(f_2 c_2)$. Then this substitution

$$s_3 = a_1 e_1 \cdot a_2 e_2 \cdot c_1 f_1 \cdot c_2 f_2 \cdot \varepsilon_2 \gamma_1 \cdot \zeta_2 \gamma_2$$

is unique. If s_3 is commutative with s_2, we transform s_3 into the above by means of $a_1 e_1 \cdot a_2 \varepsilon_1 \cdot b_1 e_2 \cdot b_2 \varepsilon_2 \cdot c_1 f_1 \cdot c_2 \zeta_1 \cdot d_1 f_2 \cdot d_2 \zeta_2$. Now assume that s_3 is non-commutative with both s_1 and s_2. The letters a_2, e_2, b_1, ε_1 are fixed, so that s_3 is commutative with both t_1 and t_2, which generate D_{15}. Let

$$s_3 = (a_1 c_2) \ldots..$$

We first note that s_3 is not commutative with both s_1 and t_2. If s_3 is commutative with s_1,

$$s_3 = a_1 c_2 \cdot a_2 c_1 \cdot \varepsilon_1 \alpha_1 \cdot \varepsilon_2 \alpha_2 \cdot \zeta_1 \beta_1 \cdot \zeta_2 \beta_2.$$

Since $a_2 b_2 \cdot c_1 d_1 \cdot e_1 \varepsilon_1 \cdot e_2 \varepsilon_2 \cdot f_1 \zeta_1 \cdot f_2 \zeta_2$ transposes s_1 and t_2, we next assume that s_3 is commutative with neither s_1 nor t_2. Now s_3 fixes c_1 and d_1, which would make it commutative with s_2, an absurdity.

There are now two groups H_3 that require attention. Let us first investigate that one generated by D_{15} and

$$s_3 = a_1 e_1 \cdot a_2 e_2 \cdot c_1 f_1 \cdot c_2 f_2 \cdot \varepsilon_2 \gamma_1 \cdot \zeta_2 \gamma_2.$$

If G is doubly transitive there is in it a substitution

$$s = (a_1 c_1) \ldots.$$

similar to s_1. From the preceding discussion we know that s must have a cycle new to D_{15} and hence is commutative with every substitution of D_{15}. But

$$\{s_3,\ s = a_1 c_1 \cdot a_2 c_2 \cdot b_1 d_1 . b_2 d_2 \ldots.\}$$

is not possible. Then G is simply transitive and the transitive group H_4 is of degree 21 or more. Since the group $\{t_1, s_3 t_2 s_3\}$ fixes a_1 and has a transitive set in the four letters $d_1, f_2, \zeta_1, \gamma_2$ and another in the letters c_2, d_2, f_1, ζ_2, there are two types to which s_4 reduces. One is

$$s_4 = (a_1 d_1)(\delta_1 -)(\delta_2 -)(\delta_3 -)\ldots\ldots$$

This substitution is commutative with t_1, so that b_1 or c_1 is replaced by δ_1. But if s_4 is commutative with s_2, both b_1 and c_1 are fixed by s_1. The other type is

$$s_4 = (a_1 c_2)(\delta_1 -)(\delta_2 -)(\delta_3 -)\ldots\ldots$$

If s_4 is commutative with neither s_1 nor t_2, s_4 fixes a_2 and b_2, whence s_4 should be commutative with t_1. Obviously s_4 is not commutative with both s_1 and t_2. If it is commutative with s_1,

$$s_4 = (a_1 c_2)(a_2 c_1)(\delta_1 -)(\delta_2 -)(\delta_3 -)(--),$$

and displaces in four cycles $\varepsilon_1, \varepsilon_2, \zeta_1$ and ζ_2. Since s_3 and s_4 are not commutative, s_4 must displace $\varepsilon_2, \zeta_2, \gamma_1$ and γ_2 in the last four cycles. Both these conditions can not be satisfied. If s_4 is commutative with t_2,

$$s_4 = a_1 c_2 \cdot b_2 d_1 \cdot e_1 f_2 \ldots\ldots,$$

since s_4 is commutative with s_3 because of the cycle $(b_2 d_1)$. But from $\{s_1, s_4\}$ the four letters e_1, e_2, f_1, f_2 are in four cycles of s_4.

There is still the substitution

$$s_3 = a_1 c_2 \cdot a_2 c_1 \cdot \varepsilon_1 \alpha_1 \cdot \varepsilon_2 \alpha_2 \cdot \zeta_1 \beta_1 \cdot \zeta_2 \beta_2 .$$

The substitution s_4 may be reduced to a single type. For we may apply to H_3 and s_4 all the transformations of

$$\{t_1,\ s_3 s_2 s_3 = b_1 c_2 \cdot d_1 a_2 \cdot e_1 \alpha_1 \cdot e_2 \alpha_2 \cdot f_1 \beta_1 \cdot f_2 \beta_2\}$$

and

$$\{e_1 e_2 \cdot f_1 f_2 \cdot \varepsilon_1 \varepsilon_2 \cdot \zeta_1 \zeta_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2,\quad e_1 f_1 \cdot e_2 f_2 \cdot \varepsilon_1 \zeta_1 \cdot \varepsilon_2 \zeta_2 \cdot \alpha_1 \beta_1 \cdot \alpha_2 \beta_2\}.$$

Then

$$s_4 = (a_1 e_1)\ldots\ldots$$

This substitution must have a cycle new to D_{15} and is in consequence commutative with every substitution of D_{15}. But such a substitution is at once seen to be out of the question.

The only non-Abelian group now remaining in our list is the D_{23} of order 6.

D_4 and D_5 may be advantageously discussed together. There are two types of substitutions to which s_3 may be reduced: $(a_1 c_1)\ldots.$ and $(a_1 e_1)\ldots..$ Now $(a_1 c_1)(a_2 c_2)\ldots.$ is not consistent with s_2; and $(a_1 e_1)(a_2 e_2)\ldots.$, since $s_2 s_3$ is of order 3, must displace one of the letters c_1, c_2 and not the other.

Consider the three groups D_i $(i = 7, 8, 9)$. A substitution s similar to s_1 can not replace one of the letters common to s_1 and s_2 by another letter of D_i. Nor can s replace one of the letters of D_i not in the common cycles by a letter new to D_i. The first remark is obviously true. In regard to the second, if s replaces α_1, say, by x, s_3 replaces one letter of each of the common cycles by a letter new to D_i, and displaces one letter from every other cycle of s_1 and s_2. Now this makes s of degree 13. It follows that substitutions of the series $s_1, s_2, \ldots.$ do not with D_i generate a transitive group.

The third group of our list can also be rejected, as will now be shown. All the substitutions of the group

$$\{a_2 b_1 \cdot c_2 d_1 \cdot e_1 \alpha_1 \cdot e_2 \alpha_2 \cdot f_1 \beta_1 \cdot f_2 \beta_2,\ \ c_1 c_2 \cdot d_1 d_2,\ \ e_1 f_1 \cdot e_2 f_2,\ \ \alpha_1 \beta_1 \cdot \alpha_2 \beta_2,\ \ e_1 e_2,\ \ f_1 f_2,\ \ \alpha_1 \alpha_2,\ .\ c_1 d_1 \cdot c_2 d_2\}$$

transform D_3 into itself. Then s_3 is either $(a_1 c_1)$ or $(a_1 e_1)\ldots..$ First, the substitution

$$s_3 = a_1 e_1 \cdot a_2 e_2 \cdot c_1 f_1 \cdot c_2 f_2 \cdot \alpha_1 \alpha_3 \cdot \beta_1 \beta_3$$

is unique. Passing at once to s_4, we have for it four possibilities:

$$(a_1 c_1)\ldots.,\quad (a_1 d_1)\ldots.,\quad (a_1 \alpha_1)\ldots.,\quad (a_1 \alpha_2)\ldots..$$

Of these, the first gives $a_1 c_1 \cdot a_2 c_2 \cdot b_1 d_1 \cdot b_2 d_2 \cdot e_1 f_1 \cdot e_2 f_2$, which is here not possible. We conclude from this that G is not doubly transitive. The other three forms of s_4 give only

$$\begin{aligned} s_4' &= a_1 d_1 \cdot a_2 d_2 \cdot b_1 c_1 \cdot b_2 c_2 \cdot \alpha_3 \alpha_4 \cdot \beta_3 \beta_4, \\ s_4'' &= a_1 \alpha_1 \cdot b_1 \alpha_2\ .\ e_1 \alpha_3 \cdot c_1 \beta_1 \cdot d_1 \beta_2 \cdot f_1 \beta_3, \\ s_4''' &= a_1 \alpha_2 \cdot b_1 \alpha_1 \cdot c_1 \beta_2 \cdot d_1 \beta_1 \cdot c_2 \alpha_4 \cdot f_2 \beta_4. \end{aligned}$$

To dispose of this case it will be sufficient to show that all the above substitutions s_4', s_4'' and s_4''' lead to transitive groups of degree less than 21. The substitution s_5 to be adjoined to $H_4' = \{H_3, s_4'\}$ is

$$a_1 \alpha_1 \cdot b_1 \alpha_2 \cdot e_1 \alpha_3 \cdot c_2 \beta_1 \cdot d_2 \beta_2 \cdot f_2 \beta_3$$

or

$$a_1 \alpha_2 \cdot b_1 \alpha_1 \cdot c_2 \beta_2 \cdot d_2 \beta_1 \cdot e_2 \beta_4 \cdot f_1 \alpha_4;$$

and in both cases H' is of degree 20 and transitive. In connection with $H_4'' = \{H_3, s_4''\}$, there is only

$$a_1 d_2 \cdot a_2 d_1 \cdot b_1 c_2 \cdot b_2 c_1 \cdot \alpha_3 \alpha_4 \cdot \beta_3 \beta_4,$$

which likewise gives a transitive group of degree 20. To $H_4''' = \{H_3, s_4'''\}$ there is the single substitution

$$a_1 d_2 \cdot a_2 d_1 \cdot b_1 c_2 \cdot b_2 c_1 \cdot \alpha_3 \beta_4 \cdot \beta_3 \alpha_4,$$

making H_4''' transitive of degree 20. The present result may be stated thus:

There is no substitution s, similar to s_1, which replaces one of the letters $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ by one of the letters $e_1, e_2, f_1, f_2, \alpha_1, \alpha_2, \beta_1, \beta_2$. Such a substitution could be transformed, by a substitution which leaves D_3 invariant, into $(a_1 e_1) \ldots\ldots$

In the second place, let

$$s_3' = a_1 c_1 \cdot a_2 c_2 \cdot b_1 d_1 \cdot b_2 d_2 \cdot \gamma_1 \gamma_2 \cdot \delta_1 \delta_2 .$$

Now s_4 must replace a_1 by γ_1, say. But $s_4 = (a_1 \gamma_1) \ldots\ldots$ bears the same relation to $\{s_1, s_3'\}$ as $(a_1 e_1) \ldots\ldots$ bears to $\{s_1, s_2\}$. Hence the group D_3 may also be struck from our list. There remain only D_6, D_{10} and D_{23}.

The substitutions of order 2 in D_{23} are

$$s_1 = a_1 a_2 \cdot b_1 b_2 \cdot c_1 c_2 \cdot d_1 d_2 \cdot e_1 e_2 \cdot f_1 f_2,$$
$$s_2 = a_1 a_3 \cdot b_1 b_3 \cdot c_1 c_3 \cdot d_1 d_3 \cdot e_1 e_3 \cdot f_1 f_3,$$
$$t_1 = a_2 a_3 \cdot b_2 b_3 \cdot c_2 c_3 \cdot d_2 d_3 \,.\, e_2 e_3 \cdot f_2 f_3 .$$

There is in G a substitution s_3 which is non-commutative with two of these substitutions, with s_1 and s_2, say. Now s_3 may or may not join two sets of D_{23}. First assume that sets are connected:

$$s_3 = a_2 b_3 \cdot a_3 b_2 \cdot c_1 c_4 \cdot d_1 d_4 \cdot e_1 e_4 \cdot f_1 f_4 .$$

In H_3 are also present

$$t_2 = a_1 b_3 \cdot a_3 b_1 \cdot c_2 c_4 \cdot d_2 d_4 \cdot e_2 e_4 \cdot f_2 f_4,$$
$$t_3 = a_1 b_2 \cdot a_2 b_1 \cdot c_3 c_4 \cdot d_3 d_4 \cdot e_3 e_4 \cdot f_3 f_4 .$$

Extend H_3 by

$$s_4 = a_1 c_1 \cdot a_2 c_2 \cdot a_3 c_3 \cdot d_4 d_5 \cdot e_4 e_5 \cdot f_4 f_5,$$

or by

$$a_1 c_3 \cdot a_3 c_1 \cdot b_2 c_4 \cdot d_2 d_5 \cdot e_2 e_5 \cdot f_2 f_5 .$$

Since these two substitutions are conjugate under

$$a_2 b_2 \cdot c_1 c_3 \cdot c_2 c_4 \cdot d_1 d_3 \cdot d_2 d_4 \cdot e_1 e_3 \cdot e_2 e_4 \cdot f_1 f_3 \cdot f_2 f_4,$$

which transforms s_1, s_2, t_1 into t_3, s_2, s_3 respectively, we use the first only. Note the substitutions of H_4:

$$t_4 = a_1 c_4 \cdot b_3 c_2 \cdot b_2 c_3 \cdot d_1 d_5 \cdot e_1 e_5 \cdot f_1 f_5,$$
$$t_5 = b_3 c_1 \cdot a_2 c_4 \cdot b_1 c_3 \cdot d_2 d_5 \cdot e_2 e_5 \cdot f_2 f_5,$$
$$t_6 = b_2 c_1 \cdot b_1 c_2 \cdot a_3 c_4 \cdot d_3 d_5 \cdot e_3 e_5 \cdot f_3 f_5.$$

Next we have

$$s_5 = a_1 d_3 \cdot a_3 d_1 \cdot b_2 d_4 \cdot c_2 \lambda \cdot e_2 f_5 \cdot f_2 e_5,$$

or

$$s_5' = a_1 d_1 \cdot a_2 d_2 \cdot a_3 d_3 \cdot c_4 d_5 \cdot e_4 e_6 \cdot f_4 f_6.$$

The group $H_5 = \{H_4, s_5\}$ has two constituents, the first of which is the $G_{1920}^{16,8}$, and the second is an imprimitive group, having five systems of two letters each permuted according to the $G_{5!}^5$. The head of the second constituent is obviously

$$F \equiv \{e_1 f_1 \cdot e_2 f_2, \quad e_1 f_1 \cdot e_3 f_3, \quad e_1 f_1 \cdot e_4 f_4, \quad e_1 f_1 \cdot e_5 f_5\}$$

of order 16. The intransitive subgroup

$$\Lambda \equiv \{e_1 e_2 \cdot f_1 f_2, \quad e_1 e_3 \cdot f_1 f_3, \quad e_1 e_4 \cdot f_1 f_4, \quad e_1 e_5 \cdot f_1 f_5\}$$

is maximal, so that this group certainly has a primitive representation of order 1920 on 16 letters. For the moment let the symbols (they are not substitutions)

1, 12, 13, 14, 15, 23, 24, 25, 34, 35, 45, 1234, 1235, 1245, 1345, 2345

represent the substitutions of the Abelian group F. If now the group Λ is transformed by these 16 substitutions in the order indicated, and if with these 16 subgroups the letters

$$\lambda, b_3, b_2, a_1, c_1, b_1, a_2, c_2, a_3, c_3, c_4, d_5, d_4, d_3, d_2, d_1$$

are associated, the generators

$$e_1 e_2 \cdot f_1 f_2, \quad e_1 e_3 \cdot f_1 f_3, \quad e_1 e_4 \cdot f_1 f_4, \quad e_4 e_5, \quad f_4 f_5, \quad e_2 f_5 \cdot e_5 f_2$$

permute the 16 subgroups of order 120 according to the substitutions

$$a_1 a_2 \cdot b_1 b_2 \cdot c_1 c_2 \cdot d_1 d_2, \quad a_1 a_3 \cdot b_1 b_3 \cdot c_1 c_3 \cdot d_1 d_3, \quad a_2 b_3 \cdot a_3 b_2 \cdot c_1 c_4 \cdot d_1 d_4,$$
$$a_1 c_1 \cdot a_2 c_2 \cdot a_3 c_3 \cdot d_1 d_5, \quad a_1 d_3 \cdot a_3 d_1 \cdot c_2 \lambda \cdot b_2 d_4,$$

respectively. Hence the intransitive group H_5 is exactly of order 1920.

Continuing the study of this group H_5 of order 1920, we write out for reference

$$t_7 = a_2 d_3 \cdot a_1 d_2 \cdot b_3 d_4 \cdot c_3 \lambda \cdot e_3 f_5 \cdot e_5 f_3, \quad t_8 = b_1 d_1 \cdot b_2 d_2 \cdot b_3 d_3 \cdot c_4 \lambda \cdot e_4 f_5 \cdot e_5 f_4,$$
$$t_9 = a_2 d_3 \cdot a_3 d_2 \cdot b_1 d_4 \cdot c_1 \lambda \cdot e_1 f_5 \cdot e_5 f_1, \quad t_{10} = c_1 d_2 \cdot c_2 d_1 \cdot b_3 d_5 \cdot a_3 \lambda \cdot e_3 f_4 \cdot e_4 f_3,$$
$$t_{11} = c_1 d_3 \cdot c_3 d_1 \cdot b_2 d_5 \cdot a_2 \lambda \cdot e_2 f_4 \cdot e_4 f_2, \quad t_{12} = c_2 d_3 \cdot c_3 d_2 \cdot b_1 d_5 \cdot a_1 \lambda \cdot e_1 f_4 \cdot e_4 f_1,$$
$$t_{13} = c_2 d_4 \cdot c_4 d_2 \cdot a_2 d_5 \cdot b_2 \lambda \cdot e_1 f_3 \cdot e_3 f_1, \quad t_{14} = c_1 d_4 \cdot c_4 d_1 \cdot a_1 d_5 \cdot b_1 \lambda \cdot e_2 f_3 \cdot e_3 f_2,$$
$$t_{15} = c_3 d_4 \cdot c_4 d_3 \cdot a_3 d_5 \cdot b_3 \lambda \cdot e_2 f_1 \cdot e_1 f_2.$$

The only substitution uniting the two sets of H_5 that need be considered is

$$s_6 = \lambda e_4 \cdot a_1 f_1 \cdot a_2 f_2 \cdot a_3 f_3 \cdot c_4 f_5 \cdot d_4 \mu,$$

which with H_5 generates a primitive group of degree 27. That in H_6 the subgroup leaving λ fixed is actually H_5 is a consequence of the equations

$$\begin{aligned} s_6 s_1 s_6 &= s_1, & s_6 s_3 s_6 &= s_3 s_6 s_3, \\ s_6 s_2 s_6 &= s_2, & s_6 s_4 s_6 &= s_4 s_6 s_4, \\ & & s_6 s_5 s_6 &= s_5 s_6 s_5, \end{aligned}$$

by means of which any substitution of $\{H_5, s_6\}$ can be put in the form

$$V_1 \text{ or } V_2 s_6 V_3,$$

where V_1, V_2, V_3 are substitutions of H_5.*

Can a larger group including H_6 contain other substitutions similar to s_1? H_6 does not transform such a substitution into one of its own members. Hence we have only to consider $(\lambda d_5) \ldots$, say. This gives uniquely (using $s_5 t_{13} s_6 t_{13} s_5 = (\lambda \mu)(d_1 e_1)(d_2 e_2)(d_3 e_3)(d_4 e_4)(d_5 e_5)$)

$$s_7 = \lambda d_5 \cdot a_1 b_1 \cdot a_2 b_2 \cdot a_3 b_3 \cdot e_5 \mu \,.\, f_5 \nu,$$

which does actually with H_6 generate a doubly transitive group of degree 28 and class 12. There remains only the question whether H_6 or H_7 may be contained invariantly in larger groups of degree 27 or 28.

It is enough to consider H_6. Let G_1 be the subgroup of G^{27} that leaves the letter μ fixed. It is assumed that H_6 is invariant in G, and in consequence H_5 is invariant in G_1. Then G_1 is not transitive but has the same two sets of letters as H_5. But the constituent of degree 16 in H_5 is not invariant in a larger group of degree 16.

We return to the group

$$\{H_4, s_5' = a_1 d_1 \cdot a_2 d_2 \cdot a_3 d_3 \cdot c_4 d_5 \cdot e_4 e_6 \cdot f_4 f_6\}.$$

The constituents of H_5' are the $G_{720}^{15,8}$ and the two $G_{6!}^{6}$. For use in the following steps we note

$$\begin{aligned} t_7' &= a_1 d_4 \cdot b_3 d_2 \cdot b_2 d_3 \cdot c_1 d_5 \cdot e_1 e_6 \cdot f_1 f_6, & t_8' &= b_3 d_1 \cdot a_2 d_4 \cdot b_1 d_3 \cdot c_2 d_5 \cdot e_2 e_6 \cdot f_2 f_6, \\ t_9' &= b_2 d_1 \cdot b_1 d_2 \cdot a_3 d_4 \cdot c_3 d_5 \cdot e_3 e_6 \cdot f_3 f_6, & t_{10}' &= c_1 d_1 \cdot c_2 d_2 \cdot c_3 d_3 \cdot c_4 d_4 \cdot e_5 e_6 \cdot f_5 f_6. \end{aligned}$$

Next we have

$$s_6' = a_1 e_1 \cdot a_2 e_2 \cdot a_3 e_3 \cdot c_4 e_5 \cdot d_4 e_6 \cdot f_4 f_7,$$

or

$$s = a_1 e_3 \cdot a_3 e_1 \cdot b_2 e_4 \cdot c_2 f_6 \cdot d_2 f_5 \cdot d_5 f_2.$$

* Cf. JORDAN, "Traité des Substitutions," 1870, p. 30.

The substitution s gives a transitive group whose generators taken in order correspond to

$$s_1,\ s_2,\ s_3,\ s_4,\ s_6,\ s_5;$$

in fact these substitutions are transformed into

$$s_1,\ s_2,\ s_3,\ s_4,\ s_5',\ s,$$

respectively, by

$$d_1 e_1 f_1 \cdot d_2 e_2 f_2 \cdot d_3 e_3 f_3 \cdot d_4 e_4 f_4 \cdot d_5 e_5 f_5 \cdot e_6 \lambda f_6 \mu.$$

Proceeding with s_6', we note

$$t_{11}' = a_1 e_4 \cdot b_3 e_2 \cdot b_2 e_3 \cdot c_1 c_5 \cdot d_1 e_6 \cdot f_1 f_7, \quad t_{12}' = b_3 e_1 \cdot a_2 e_4 \cdot b_1 e_3 \cdot c_2 e_5 \cdot d_2 e_6 \cdot f_2 f_7,$$
$$t_{13}' = b_2 e_1 \cdot b_1 e_2 \cdot a_3 e_4 \cdot c_3 e_5 \cdot d_3 e_6 \cdot f_3 f_7, \quad t_{14}' = c_1 e_1 \cdot c_2 e_2 \cdot c_3 e_3 \cdot c_4 e_4 \cdot d_5 e_6 \cdot f_5 f_7,$$
$$t_{15}' = d_1 e_1 \cdot d_2 e_2 \cdot d_3 e_3 \cdot d_4 e_4 \cdot d_5 e_5 \cdot f_6 f_7.$$

Now we have

$$s_7' = a_1 f_1 \cdot a_2 f_2 \cdot a_3 f_3 \cdot c_4 f_5 \cdot d_4 f_6 \cdot e_4 f_7$$

or

$$s_7'' = a_1 f_3 \cdot a_3 f_1 \cdot b_2 f_4 \cdot c_2 e_6 \cdot d_2 e_5 \cdot e_2 d_5.$$

Since $d_1 f_1 \cdot d_2 f_2 \cdot d_3 f_3 \cdot d_4 f_4 \cdot d_5 f_5 \cdot \lambda e_6 \cdot \mu f_6$ is commutative with s_1, s_2, s_3, s_4, and transforms s_5 and s_6 into s_7'' and s_5' respectively, only s_7' is to be considered. The group $\{H_5', s_6' s_7' s_6' = e_1 f_1 \cdot e_2 f_2 \cdot e_3 f_3 \cdot e_4 f_4 \cdot e_5 f_5 \cdot e_6 f_6\}$ has two transitive constituents, the primitive $G_{720}^{15,8}$ and the imprimitive $G_{2(6!)}^{10}$. The relations

$$s_7' s_1 s_7' = s_1, \quad s_7' s_3 s_7' = s_3 s_7' s_3, \quad s_7' s_5' s_7' = s_5' s_7' s_5',$$
$$s_7' s_2 s_7' = s_2, \quad s_7' s_4 s_7' = s_4 s_7' s_4, \quad s_7' (s_6' s_7' s_6') s_7' = s_6' s_7' s_6' s_7' s_6' s_7' s_6'$$

show that the primitive group H_7' really has $\{H_5', s_6' s_7' s_6'\}$ as its subgroup leaving f_7 fixed. This group H_7' is simply isomorphic to the $G_{8!}^8$. It is not invariant in a larger group of the same degree. Then any larger group containing it has a substitution similar to s_1 new to H_7'. After transformation by the substitutions of $\{H_5', s_6' s_7' s_6'\}$, there are two types of substitutions involving f_7 which offer a chance of extending H_7': $(f_7 e_4) \ldots$ and $(f_7 c_2) \ldots$. The first, $f_7 e_4 \cdot a_1 f_1 \ldots$, is already in H_7', and the second,

$$s_8' = f_7 c_2 \cdot e_2 f_5 \cdot e_5 f_2 \cdot a_1 d_3 \cdot b_2 d_4 \cdot d_1 a_3,$$

is the substitution s_5 already discussed.

We return to D_{23}, and assume that s_3 does not join sets of D_{23}. Then

$$s_3 = a_1 a_4 \cdot b_1 b_4 \cdot c_1 c_4 \cdot d_1 d_4 \cdot e_1 e_4 \cdot f_1 f_4,$$
$$t_2 = a_2 a_4 \cdot b_2 b_4 \cdot c_2 c_4 \cdot d_2 d_4 \cdot e_2 e_4 \cdot f_2 f_4,$$
$$t_3 = a_3 a_4 \cdot b_3 b_4 \cdot c_3 c_4 \cdot d_3 d_4 \cdot e_3 e_4 \cdot f_3 f_4.$$

There exists a substitution s_4 which connects the above letters with new letters, and since we have discussed the case where s_4 connects sets of such a group as $\{s_1, s_2\}$ and is not commutative with all its substitutions, we have uniquely

$$s_4 = a_1 a_5 \cdot b_1 b_5 \cdot c_1 c_5 \cdot d_1 d_5 \cdot e_1 e_5 \cdot f_1 f_5,$$

and similarly,

$$s_5 = a_1 a_6 \cdot b_1 b_6 \cdot c_1 c_6 \cdot d_1 d_6 \cdot e_1 e_6 \cdot f_1 f_6.$$

Now there exists a substitution s_6 that connects two sets of H_5 and has no cycle of new letters:

$$s_6 = a_1 b_1 \cdot a_2 b_2 \cdot a_3 b_3 \cdot a_4 b_4 \cdot a_5 b_5 \cdot a_6 b_6,$$

uniquely, and so on,

$$s_7 = a_1 c_1 \cdot a_2 c_2 \cdot a_3 c_3 \cdot a_4 c_4 \cdot a_5 c_5 \cdot a_6 c_6,$$
$$s_8 = a_1 d_1 \cdot a_2 d_2 \cdot a_3 d_3 \cdot a_4 d_4 \cdot a_5 d_5 \cdot a_6 d_6,$$
$$s_9 = a_1 e_1 \cdot a_2 e_2 \cdot a_3 e_3 \cdot a_4 e_4 \cdot a_5 e_5 \cdot a_6 e_6,$$
$$s_{10} = a_1 f_1 \cdot a_2 f_2 \cdot a_3 f_3 \cdot a_4 f_4 \cdot a_5 f_5 \cdot a_6 f_6.$$

The transitive group H_{10} is imprimitive of degree 36 and order $(6\,!)^2$. It is contained in one and only one primitive group $G^{36,\,12}_{2(6\,!)^2}$.

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